The Rokhlin dimension of topological $\mathbb{Z}^m$-actions

The structure and classification of nuclear C*-algebras

Gábor Szabó

WWU Münster

April 2013

Today, I would like to convince you of the following:

Theorem

Let X be a compact metric space of finite covering dimension and let $\alpha : \mathbb{Z}^m \curvearrowright X$ be a free continuous group action.

Today, I would like to convince you of the following:

Theorem

Let X be a compact metric space of finite covering dimension and let $\alpha : \mathbb{Z}^m \curvearrowright X$ be a free continuous group action. Then the transformation group C-algebra $\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}^m$ has finite nuclear dimension.*

Today, I would like to convince you of the following:

Theorem

Let X be a compact metric space of finite covering dimension and let $\alpha : \mathbb{Z}^m \curvearrowright X$ be a free continuous group action. Then the transformation group C-algebra $\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}^m$ has finite nuclear dimension.*

In particular, when α is assumed to be free and minimal, then $\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}^m$ is $\mathcal{Z}$-stable.

Notation

We will use the following notations:

- X is a compact metric space that is (mostly) assumed to have finite covering dimension.
- A is a unital C*-algebra.
- Either $\alpha : \mathbb{Z}^m \curvearrowright A$ is a group action via automorphisms or $\alpha : \mathbb{Z}^m \curvearrowright X$ is a continuous group action on X. In the topological case, α is usually assumed to be free.
- If M is some set and $F \subset M$ is a finite subset, we write $F \Subset M$.
- For $n \in \mathbb{N}$, let
$$B_n^m = \{0, \dots, n-1\}^m \subset \mathbb{Z}^m.$$
If m is known from context, we write B_n instead.

Definition (Hirshberg-Winter-Zacharias)

Let A be a unital C*-algebra, and let $\alpha : \mathbb{Z}^m \curvearrowright A$ be a group action via automorphisms. We say that the action α has (cyclic) Rokhlin dimension d, and write $\dim_{\mathsf{Rok}}^{\mathsf{cyc}}(\alpha) = d$, if d is the smallest natural number with the following property:

Definition (Hirshberg-Winter-Zacharias)

Let A be a unital C*-algebra, and let $\alpha : \mathbb{Z}^m \curvearrowright A$ be a group action via automorphisms. We say that the action α has (cyclic) Rokhlin dimension d, and write $\dim_{\mathrm{Rok}}^{\mathrm{cyc}}(\alpha) = d$, if d is the smallest natural number with the following property:

For all $F \Subset A, \varepsilon > 0, n \in \mathbb{N}$, there exist positive contractions $(f_v^{(l)})_{v \in B_n}^{l=0,\dots,d}$ in A satisfying the following properties:

Definition (Hirshberg-Winter-Zacharias)

Let A be a unital C*-algebra, and let $\alpha : \mathbb{Z}^m \curvearrowright A$ be a group action via automorphisms. We say that the action α has (cyclic) Rokhlin dimension d, and write $\dim_{\mathrm{Rok}}^{\mathrm{cyc}}(\alpha) = d$, if d is the smallest natural number with the following property:

For all $F \Subset A, \varepsilon > 0, n \in \mathbb{N}$, there exist positive contractions $(f_v^{(l)})_{v \in B_n}^{l=0,\dots,d}$ in A satisfying the following properties:

(1) $\|\mathbf{1}_A - \sum_{l=0}^{d} \sum_{v \in B_n} f_v^{(l)}\| \leq \varepsilon.$

If there is no such d, we write $\dim_{\mathrm{Rok}}^{\mathrm{cyc}}(\alpha) = \infty$.

Definition (Hirshberg-Winter-Zacharias)

Let A be a unital C*-algebra, and let $\alpha : \mathbb{Z}^m \curvearrowright A$ be a group action via automorphisms. We say that the action α has (cyclic) Rokhlin dimension d, and write $\dim_{\text{Rok}}^{\text{cyc}}(\alpha) = d$, if d is the smallest natural number with the following property:
For all $F \Subset A, \varepsilon > 0, n \in \mathbb{N}$, there exist positive contractions $(f_v^{(l)})_{v\in B_n}^{l=0,\dots,d}$ in A satisfying the following properties:

(1) $\|\mathbf{1}_A - \sum_{l=0}^{d} \sum_{v\in B_n} f_v^{(l)}\| \leq \varepsilon$.

(2) $\|f_v^{(l)} f_w^{(l)}\| \leq \varepsilon$ for all $l = 0, \dots, d$ and $v \neq w$ in B_n.

If there is no such d, we write $\dim_{\text{Rok}}^{\text{cyc}}(\alpha) = \infty$.

Definition (Hirshberg-Winter-Zacharias)

Let A be a unital C*-algebra, and let $\alpha : \mathbb{Z}^m \curvearrowright A$ be a group action via automorphisms. We say that the action α has (cyclic) Rokhlin dimension d, and write $\dim_{\text{Rok}}^{\text{cyc}}(\alpha) = d$, if d is the smallest natural number with the following property:

For all $F \Subset A, \varepsilon > 0, n \in \mathbb{N}$, there exist positive contractions $(f_v^{(l)})_{v \in B_n}^{l=0,\dots,d}$ in A satisfying the following properties:

(1) $\|\mathbf{1}_A - \sum_{l=0}^{d} \sum_{v \in B_n} f_v^{(l)}\| \leq \varepsilon.$

(2) $\|f_v^{(l)} f_w^{(l)}\| \leq \varepsilon$ for all $l = 0, \dots, d$ and $v \neq w$ in B_n.

(3) $\|\alpha^v(f_w^{(l)}) - f_{v+w}^{(l)}\| \leq \varepsilon$ for all $l = 0, \dots, d$ and $v, w \in B_n$. (!)

If there is no such d, we write $\dim_{\text{Rok}}^{\text{cyc}}(\alpha) = \infty$.

Definition (Hirshberg-Winter-Zacharias)

Let A be a unital C*-algebra, and let $\alpha : \mathbb{Z}^m \curvearrowright A$ be a group action via automorphisms. We say that the action α has (cyclic) Rokhlin dimension d, and write $\dim_{\text{Rok}}^{\text{cyc}}(\alpha) = d$, if d is the smallest natural number with the following property:

For all $F \Subset A, \varepsilon > 0, n \in \mathbb{N}$, there exist positive contractions $(f_v^{(l)})_{v \in B_n}^{l=0,\dots,d}$ in A satisfying the following properties:

(1) $\|\mathbf{1}_A - \sum_{l=0}^{d} \sum_{v \in B_n} f_v^{(l)}\| \leq \varepsilon$.

(2) $\|f_v^{(l)} f_w^{(l)}\| \leq \varepsilon$ for all $l = 0, \dots, d$ and $v \neq w$ in B_n.

(3) $\|\alpha^v(f_w^{(l)}) - f_{v+w}^{(l)}\| \leq \varepsilon$ for all $l = 0, \dots, d$ and $v, w \in B_n$. (!)

(4) $\|[f_v^{(l)}, a]\| \leq \varepsilon$ for all $l = 0, \dots, d$, $v \in B_n$ and $a \in F$.

If there is no such d, we write $\dim_{\text{Rok}}^{\text{cyc}}(\alpha) = \infty$.

The usefulness of this notion is illustrated in the following theorem:

Theorem (Hirshberg-Winter-Zacharias 2012 for $m = 1$.)

Let A be a unital C-algebra and let $\alpha : \mathbb{Z}^m \curvearrowright A$ be a group action via automorphisms. Then*

$$\dim_{nuc}(A \rtimes_\alpha \mathbb{Z}^m) \leq 2^m(\dim_{nuc}(A) + 1)(\dim_{Rok}^{cyc}(\alpha) + 1) - 1.$$

Definition (Winter)

Let $(X, \alpha, \mathbb{Z}^m)$ be a topological dynamical system. We say that α has Rokhlin dimension d, and write $\dim_{\mathsf{Rok}}(\alpha) = d$, if d is the smallest natural number with the following property:

Definition (Winter)

Let $(X, \alpha, \mathbb{Z}^m)$ be a topological dynamical system. We say that α has Rokhlin dimension d, and write $\dim_{\mathsf{Rok}}(\alpha) = d$, if d is the smallest natural number with the following property:
For all $n \in \mathbb{N}$, there exists a family of open sets

$$\mathcal{R} = \left\{ U_v^{(l)} \mid l = 0, \ldots, d, \; v \in B_n \right\}$$

in X (we call this a Rokhlin cover) such that

Definition (Winter)

Let $(X, \alpha, \mathbb{Z}^m)$ be a topological dynamical system. We say that α has Rokhlin dimension d, and write $\dim_{\mathsf{Rok}}(\alpha) = d$, if d is the smallest natural number with the following property:
For all $n \in \mathbb{N}$, there exists a family of open sets

$$\mathcal{R} = \left\{ U_v^{(l)} \mid l = 0, \ldots, d, \ v \in B_n \right\}$$

in X (we call this a Rokhlin cover) such that

- $U_v^{(l)} = \alpha^v(U_0^{(l)})$ for all $l = 0, \ldots, d$ and $v \in B_n$.

If there is no such d, then $\dim_{\mathsf{Rok}}(\alpha) = \infty$.

Definition (Winter)

Let $(X, \alpha, \mathbb{Z}^m)$ be a topological dynamical system. We say that α has Rokhlin dimension d, and write $\dim_{\mathsf{Rok}}(\alpha) = d$, if d is the smallest natural number with the following property:
For all $n \in \mathbb{N}$, there exists a family of open sets

$$\mathcal{R} = \left\{ U_v^{(l)} \mid l = 0, \ldots, d,\ v \in B_n \right\}$$

in X (we call this a Rokhlin cover) such that

- $U_v^{(l)} = \alpha^v(U_0^{(l)})$ for all $l = 0, \ldots, d$ and $v \in B_n$.
- For all l, the sets $\left\{ U_v^{(l)} \mid v \in B_n \right\}$ are pairwise disjoint.

If there is no such d, then $\dim_{\mathsf{Rok}}(\alpha) = \infty$.

Definition (Winter)

Let $(X, \alpha, \mathbb{Z}^m)$ be a topological dynamical system. We say that α has Rokhlin dimension d, and write $\dim_{\mathsf{Rok}}(\alpha) = d$, if d is the smallest natural number with the following property:
For all $n \in \mathbb{N}$, there exists a family of open sets

$$\mathcal{R} = \left\{ U_v^{(l)} \mid l = 0, \ldots, d, \ v \in B_n \right\}$$

in X (we call this a Rokhlin cover) such that

- $U_v^{(l)} = \alpha^v(U_0^{(l)})$ for all $l = 0, \ldots, d$ and $v \in B_n$.
- For all l, the sets $\left\{ U_v^{(l)} \mid v \in B_n \right\}$ are pairwise disjoint.
- $\mathcal{R}$ is an open cover of X.

If there is no such d, then $\dim_{\mathsf{Rok}}(\alpha) = \infty$.

Definition (Winter)

Let $(X, \alpha, \mathbb{Z}^m)$ be a topological dynamical system. We say that α has dynamic dimension d, and write $\dim_{\text{dyn}}(\alpha) = d$, if d is the smallest natural number with the following property:
For all $n \in \mathbb{N}$ and all open covers $\mathcal{U}$ of X, there exists a family of open sets

$$\mathcal{R} = \left\{ U_{i,v}^{(l)} \mid l = 0, \ldots, d, \ v \in B_n, i = 1, \ldots, K(l) \right\}$$

in X (we call this a Rokhlin cover) such that

- $U_{i,v}^{(l)} = \alpha^v(U_{i,0}^{(l)})$ for all $l = 0, \ldots, d$, $i \leq K(l)$ and $v \in B_n$.
- For all l, the sets $\left\{ U_{i,v}^{(l)} \mid v \in B_n, i \leq K(l) \right\}$ are pairwise disjoint.
- $\mathcal{R}$ is an open cover of X that refines $\mathcal{U}$.

If there is no such d, then $\dim_{\text{dyn}}(\alpha) = \infty$.

To set the C*-algebraic Rokhlin dimension in relation to this topological business, the following fact is key:

Lemma

Let $\alpha : \mathbb{Z}^m \curvearrowright X$ be a continuous group action on a compact metric space. Let $\bar{\alpha} : \mathbb{Z}^m \curvearrowright \mathcal{C}(X)$ be the induced C-algebraic action. Then*

$$\dim_{Rok}^{cyc}(\bar{\alpha}) \leq 2^m(\dim_{Rok}(\alpha) + 1) - 1.$$

The following is one of the few known results concerning the finiteness of Rokhlin dimension in concrete cases:

The following is one of the few known results concerning the finiteness of Rokhlin dimension in concrete cases:

Theorem (Hirshberg-Winter-Zacharias 2012)

For a minimal homeomorphism $\varphi : X \to X$ on an infinite space, we have

$$\dim_{Rok}(\varphi) \leq 2(\dim(X) + 1) - 1$$

and

$$\dim_{dyn}(\varphi) \leq 2(\dim(X) + 1)^2 - 1.$$

The following is one of the few known results concerning the finiteness of Rokhlin dimension in concrete cases:

Theorem (Hirshberg-Winter-Zacharias 2012)

For a minimal homeomorphism $\varphi : X \to X$ on an infinite space, we have

$$\dim_{Rok}(\varphi) \leq 2(\dim(X) + 1) - 1$$

and

$$\dim_{dyn}(\varphi) \leq 2(\dim(X) + 1)^2 - 1.$$

Although the statement is purely topological, the proof made heavy use of the structure of the orbit-breaking algebras

$$A_Y = C^*(\mathcal{C}(X) \cup u \cdot \mathcal{C}_0(X \setminus Y)) \subset \mathcal{C}(X) \rtimes_\varphi \mathbb{Z} \quad \text{for } Y = \overline{Y} \subset X.$$

Almost simultaniously, Yonathan Gutman published topological results that enabled a purely topological and easy proof of this theorem.

Almost simultaniously, Yonathan Gutman published topological results that enabled a purely topological and easy proof of this theorem. It uses the so called marker property for aperiodic homeomorphisms.

Almost simultaniously, Yonathan Gutman published topological results that enabled a purely topological and easy proof of this theorem. It uses the so called marker property for aperiodic homeomorphisms.

Definition (Markers)

Let $\alpha : G \curvearrowright X$ be a continuous group action. Let $F \Subset G$. An F-marker is an open set $Z \subset X$ with

- $\alpha^g(\overline{Z}) \cap \alpha^h(\overline{Z}) = \emptyset$ for all $g \neq h$ in F.
- $X = \bigcup_{g \in G} \alpha^g(Z)$.

Almost simultaniously, Yonathan Gutman published topological results that enabled a purely topological and easy proof of this theorem. It uses the so called marker property for aperiodic homeomorphisms.

Definition (Markers)

Let $\alpha : G \curvearrowright X$ be a continuous group action. Let $F \Subset G$. An F-marker is an open set $Z \subset X$ with

- $\alpha^g(\overline{Z}) \cap \alpha^h(\overline{Z}) = \emptyset$ for all $g \neq h$ in F.
- $X = \displaystyle\bigcup_{g \in G} \alpha^g(Z)$.

We say that α has the marker property if there exist F-markers for all $F \Subset G$.

Definition (Controlled markers)

Let $\alpha : \mathbb{Z}^m \curvearrowright X$ be a continuous group action. Let $F \Subset \mathbb{Z}^m$ and $L \in \mathbb{N}$. An L-controlled F-marker is an open set $Z \subset X$ with

- $\alpha^v(\overline{Z}) \cap \alpha^w(\overline{Z}) = \emptyset$ for all $v \neq w$ in F.
- $X = \bigcup_{l=1}^{L} \bigcup_{v \in F} \alpha^{v_l + v}(Z)$ for some $v_1, \ldots, v_L \in \mathbb{Z}^m$.

Definition (Controlled markers)

Let $\alpha : \mathbb{Z}^m \curvearrowright X$ be a continuous group action. Let $F \Subset \mathbb{Z}^m$ and $L \in \mathbb{N}$. An L-controlled F-marker is an open set $Z \subset X$ with

- $\alpha^v(\overline{Z}) \cap \alpha^w(\overline{Z}) = \emptyset$ for all $v \neq w$ in F.
- $X = \bigcup_{l=1}^{L} \bigcup_{v \in F} \alpha^{v_l + v}(Z)$ for some $v_1, \dots, v_L \in \mathbb{Z}^m$.

We say that α has the L-controlled marker property if there exist L-controlled B_n-markers for all $n \in \mathbb{N}$.

Theorem (Gutman 2012)

If X has finite covering dimension and $\varphi : X \to X$ is aperiodic, then φ has the marker property.

Although the result is stated this way, careful reading of his proof yields something stronger:

Theorem (Gutman 2012)

Let X have finite covering dimension d and let $\varphi : X \to X$ be an aperiodic homeomorphism. For all n, there exists an n-marker (i.e. a $\{0, \ldots, n-1\}$-marker) $Z \subset X$ such that

$$X = \bigcup_{j=0}^{2(d+1)n-1} \varphi^j(Z).$$

In particular, φ has the $2(d+1)$-controlled marker property.

Corollary

For an aperiodic homeomorphism $\varphi : X \to X$, we have

$$\dim_{Rok}(\varphi) \leq 2(\dim(X) + 1) - 1$$

and

$$\dim_{dyn}(\varphi) \leq 2(\dim(X) + 1)^2 - 1.$$

Corollary

For an aperiodic homeomorphism $\varphi : X \to X$, *we have*

$$\dim_{Rok}(\varphi) \leq 2(\dim(X) + 1) - 1$$

and

$$\dim_{dyn}(\varphi) \leq 2(\dim(X) + 1)^2 - 1.$$

Proof. We prove just the first inequality. For any n, let us find an n-marker Z such that $X = \bigcup_{j=0}^{2(d+1)n-1} \varphi^j(Z)$. If we set $U_j^{(l)} = \varphi^{ln+j}(Z)$ for $l = 0, \ldots, 2(d+1) - 1$ and $j = 0, \ldots, n-1$, we get a Rokhlin cover with the desired properties.

This topological proof of finite Rokhlin dimension for aperiodic homeomorphisms is much simpler than the C*-algebraic one.

This topological proof of finite Rokhlin dimension for aperiodic homeomorphisms is much simpler than the C*-algebraic one.

Moreover, these methods do not crucially use the group structure of $\mathbb{Z}$, so this seems to be a good approach for more general group actions.

This topological proof of finite Rokhlin dimension for aperiodic homeomorphisms is much simpler than the C*-algebraic one.

Moreover, these methods do not crucially use the group structure of $\mathbb{Z}$, so this seems to be a good approach for more general group actions.

There is just one important technical obstacle that has to be tackled in order for this approach to be sensible for groups $\neq \mathbb{Z}$:

Definition

Let G be a countably infinite group and $\alpha : G \curvearrowright X$ a continuous group action. Let $M \subset G$ be a subset and $k \in \mathbb{N}$ be some natural number.

Definition

Let G be a countably infinite group and $\alpha : G \curvearrowright X$ a continuous group action. Let $M \subset G$ be a subset and $k \in \mathbb{N}$ be some natural number. We say that a set $E \subset X$ is (M, k)-disjoint, if for all distinct elements $\gamma(0), \ldots, \gamma(k) \in M$ we have

$$\alpha^{\gamma(0)}(E) \cap \cdots \cap \alpha^{\gamma(k)}(E) = \emptyset.$$

Definition

Let G be a countably infinite group and $\alpha : G \curvearrowright X$ a continuous group action. Let $M \subset G$ be a subset and $k \in \mathbb{N}$ be some natural number. We say that a set $E \subset X$ is (M,k)-disjoint, if for all distinct elements $\gamma(0), \ldots, \gamma(k) \in M$ we have

$$\alpha^{\gamma(0)}(E) \cap \cdots \cap \alpha^{\gamma(k)}(E) = \emptyset.$$

We call E topologically G-small if E is (G,k)-disjoint for some k.

Definition

Let G be a countably infinite group and $\alpha : G \curvearrowright X$ a continuous group action. Let $M \subset G$ be a subset and $k \in \mathbb{N}$ be some natural number. We say that a set $E \subset X$ is (M, k)-disjoint, if for all distinct elements $\gamma(0), \ldots, \gamma(k) \in M$ we have

$$\alpha^{\gamma(0)}(E) \cap \cdots \cap \alpha^{\gamma(k)}(E) = \emptyset.$$

We call E topologically G-small if E is (G, k)-disjoint for some k.

The system (X, α, G) is said to have the topological small boundary property (TSBP), if X has a topological base consisting of open sets U such that the boundaries ∂U are topologically G-small.

Definition

Let G be a countably infinite group and $\alpha : G \curvearrowright X$ a continuous group action. Let $M \subset G$ be a subset and $k \in \mathbb{N}$ be some natural number. We say that a set $E \subset X$ is (M,k)-disjoint, if for all distinct elements $\gamma(0), \ldots, \gamma(k) \in M$ we have

$$\alpha^{\gamma(0)}(E) \cap \cdots \cap \alpha^{\gamma(k)}(E) = \emptyset.$$

We call E topologically G-small if E is (G,k)-disjoint for some k.

The system (X, α, G) is said to have the topological small boundary property (TSBP), if X has a topological base consisting of open sets U such that the boundaries ∂U are topologically G-small.
If we can arrange that the smallness constants are bounded by some $d \in \mathbb{N}$, we say that (X, α, G) has the bounded TSBP with respect to d.

Theorem (Lindenstrauss 1995)

If X has finite covering dimension d and $\varphi : X \to X$ is an aperiodic homeomorphism, then the system (X, φ) has the bounded TSBP with respect to d.

Theorem (Lindenstrauss 1995)

If X has finite covering dimension d and $\varphi : X \to X$ is an aperiodic homeomorphism, then the system (X, φ) has the bounded TSBP with respect to d.

This property is at the heart of Gutman's proof of the (controlled) marker property of aperiodic homeomorphisms. In order to generalize Gutman's approach, one needs an analogous theorem in greater generality.

Indeed, the analogous theorem holds in much greater generality:

Theorem

Let X have finite covering dimension d and let G be a countably infinite group that acts freely via $\alpha : G \curvearrowright X$.

Indeed, the analogous theorem holds in much greater generality:

Theorem

Let X have finite covering dimension d and let G be a countably infinite group that acts freely via $\alpha : G \curvearrowright X$. Then the system (X, α, G) has the bounded TSBP with respect to d.

Indeed, the analogous theorem holds in much greater generality:

Theorem

Let X have finite covering dimension d and let G be a countably infinite group that acts freely via $\alpha : G \curvearrowright X$. Then the system (X, α, G) has the bounded TSBP with respect to d.

As it turns out, this is really a crucial building block for the proof of the following lemma, that is a generalization of Gutman's result.

Lemma

Let X have finite covering dimension d and let G be a countably infinite group that acts freely via $\alpha : G \curvearrowright X$.

Lemma

Let X have finite covering dimension d and let G be a countably infinite group that acts freely via $\alpha : G \curvearrowright X$. Let $F \Subset G$ a finite subset, and let $g_1, \dots, g_d \in G$ be elements such that

$$FF^{-1},\ g_1FF^{-1},\ \dots,\ g_dFF^{-1}$$

are pairwise disjoint.

Lemma

Let X have finite covering dimension d and let G be a countably infinite group that acts freely via $\alpha : G \curvearrowright X$. Let $F \Subset G$ a finite subset, and let $g_1, \ldots, g_d \in G$ be elements such that

$$FF^{-1},\ g_1FF^{-1},\ \ldots,\ g_dFF^{-1}$$

are pairwise disjoint. Set $M = FF^{-1} \dot\cup g_1FF^{-1} \dot\cup \ldots \dot\cup g_dFF^{-1}$.

Lemma

Let X have finite covering dimension d and let G be a countably infinite group that acts freely via $\alpha : G \curvearrowright X$. Let $F \Subset G$ a finite subset, and let $g_1, \ldots, g_d \in G$ be elements such that

$$FF^{-1},\ g_1FF^{-1},\ \ldots,\ g_dFF^{-1}$$

are pairwise disjoint. Set $M = FF^{-1} \dot\cup g_1FF^{-1} \dot\cup \ldots \dot\cup g_dFF^{-1}$.

Then there exists an F-marker Z such that $X = \bigcup_{g \in M} \alpha^g(Z)$.

A consequence for $G = \mathbb{Z}^m$ is:

Theorem

Let $\alpha : \mathbb{Z}^m \curvearrowright X$ *be a free continuous action on a compact metric space of finite covering dimension* d. *Then* $(X, \alpha, \mathbb{Z}^m)$ *has the* $2^m(d+1)$*-controlled marker property.*

A consequence for $G = \mathbb{Z}^m$ is:

Theorem

Let $\alpha : \mathbb{Z}^m \curvearrowright X$ be a free continuous action on a compact metric space of finite covering dimension d. Then $(X, \alpha, \mathbb{Z}^m)$ has the $2^m(d+1)$-controlled marker property.

Idea of the proof: $(B_n - B_n)$ is contained in a translate of B_{2n}. In $\mathbb{Z}^m$, one needs at most 2^m translates of B_n to cover B_{2n}, hence also to cover $(B_n - B_n)$.

Similarly as in the case $m = 1$, it follows that:

Corollary

For a free continuous group action $\alpha : \mathbb{Z}^m \curvearrowright X$, *we have*

$$\dim_{Rok}(\alpha) \leq 2^m(\dim(X) + 1) - 1$$

and

$$\dim_{dyn}(\alpha) \leq 2^m(\dim(X) + 1)^2 - 1.$$

Similarly as in the case $m = 1$, it follows that:

Corollary

For a free continuous group action $\alpha : \mathbb{Z}^m \curvearrowright X$, *we have*

$$\dim_{Rok}(\alpha) \leq 2^m(\dim(X) + 1) - 1$$

and

$$\dim_{dyn}(\alpha) \leq 2^m(\dim(X) + 1)^2 - 1.$$

Finally, we can combine this with the statements about the C*-algebraic Rokhlin dimension to get:

Theorem

Let X be a compact metric space of finite covering dimension and let $\alpha : \mathbb{Z}^m \curvearrowright X$ be a free continuous group action.

Theorem

Let X be a compact metric space of finite covering dimension and let $\alpha : \mathbb{Z}^m \curvearrowright X$ be a free continuous group action. Then the induced C-algebraic action $\bar{\alpha}$ on $\mathcal{C}(X)$ has finite Rokhlin dimension, and the transformation group C*-algebra $\mathcal{C}(X) \rtimes_{\bar{\alpha}} \mathbb{Z}^m$ has finite nuclear dimension.*

Theorem

Let X be a compact metric space of finite covering dimension and let $\alpha : \mathbb{Z}^m \curvearrowright X$ be a free continuous group action. Then the induced C-algebraic action $\bar{\alpha}$ on $\mathcal{C}(X)$ has finite Rokhlin dimension, and the transformation group C*-algebra $\mathcal{C}(X) \rtimes_{\bar{\alpha}} \mathbb{Z}^m$ has finite nuclear dimension. More specifically, we have*

$$\dim_{Rok}^{cyc}(\bar{\alpha}) \leq 2^{2m}(\dim(X) + 1) - 1$$

and thus

$$\dim_{nuc}(\mathcal{C}(X) \rtimes_{\bar{\alpha}} \mathbb{Z}^m) \leq 2^{3m}(\dim(X) + 1)^2 - 1.$$

Theorem

Let X be a compact metric space of finite covering dimension and let $\alpha : \mathbb{Z}^m \curvearrowright X$ be a free continuous group action. Then the induced C-algebraic action $\bar{\alpha}$ on $\mathcal{C}(X)$ has finite Rokhlin dimension, and the transformation group C*-algebra $\mathcal{C}(X) \rtimes_{\bar{\alpha}} \mathbb{Z}^m$ has finite nuclear dimension. More specifically, we have*

$$\dim_{Rok}^{cyc}(\bar{\alpha}) \leq 2^{2m}(\dim(X)+1)-1$$

and thus

$$\dim_{nuc}(\mathcal{C}(X) \rtimes_{\bar{\alpha}} \mathbb{Z}^m) \leq 2^{3m}(\dim(X)+1)^2-1.$$

In particular, when α is assumed to be free and minimal, then $\mathcal{C}(X) \rtimes_{\bar{\alpha}} \mathbb{Z}^m$ is $\mathcal{Z}$-stable.