Finite group actions and the UCT problem

Workshop on Model Theory and Operator Algebras

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1. Introduction

2. Rokhlin actions on UHF-absorbing C^*-algebras

3. Some examples

4. Finite group actions on $\mathcal{O}_2$ and the UCT

Unless specified otherwise, we will stick to the following notation throughout this talk:

- G is a finite group.
- A is a separable, unital C^*-algebra.
- α, β or γ are finite group actions on such a C^*-algebra.

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Definition (Izumi)

Let $\alpha : G \curvearrowright A$ be given, and let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter. Then α has the Rokhlin property, if there exists a unital, equivariant $*$-homomorphism

$$(\mathcal{C}(G), G\text{-shift}) \longrightarrow (A_\omega \cap A', \alpha_\omega).$$

We also call such α a Rokhlin action.

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Let A be simple, G a finite group and $\alpha : G \curvearrowright A$ a Rokhlin action. Then $K_*(A \rtimes_\alpha G)$ is isomorphic to the subgroup $\bigcap_{g\in G} \ker(\mathrm{id} - K_*(\alpha_g))$ inside $K_*(A)$.

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Theorem (Barlak-S)

Let A be given, G a finite group and $\alpha : G \curvearrowright A$ a Rokhlin action. Assume moreover that $A \cong M_{|G|^\infty} \otimes A$. Then $A \rtimes_\alpha G$ decomposes as a direct limit of matrix algebras over A, with connecting maps depending only on α.

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Example

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However, there are certain canonical examples.

Notation

Let G be a finite group. The matrix algebra $M_{|G|}$ is generated by elements $\{e_{g,h}\}_{g,h\in G}$ satisfying the relations $e_{h_1,h_2} \cdot e_{h_3,h_4} = \delta_{h_2,h_3} e_{h_1,h_4}$. One denotes

$$M_{|G|^\infty} = \bigotimes_{\mathbb{N}} M_{|G|} = \varinjlim \left\{ M_{|G|}^{\otimes n}, [x \mapsto x \otimes \mathbf{1}_{|G|}] \right\}.$$

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Example

Consider the left-regular representation $\lambda : G \to \mathcal{U}(M_{|G|})$ defined by $\lambda(g) = \sum_{h\in G} e_{gh,h}$. One obtains an induced Rokhlin action $\beta^G : G \curvearrowright M_{|G|^\infty}$ by $\beta^G_g = \bigotimes_{\mathbb{N}} \mathrm{Ad}(\lambda(g))$ for all $g \in G$.

Fact

If $A \cong M_{|G|^\infty} \otimes A$, *then the canonical embedding* $A \longrightarrow M_{|G|^\infty} \otimes A$ *given by* $x \mapsto \mathbf{1} \otimes x$ *is approximately unitarily equivalent to an isomorphism.*

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Example

Let us assume that $A \cong M_{|G|^\infty} \otimes A$. Let $\alpha : G \curvearrowright A$ be any action. Then $\beta^G \otimes \alpha$ is an action with the Rokhlin property on $M_{|G|^\infty} \otimes A$. Identifying this with A in the above way, this yields a Rokhlin action on A that is pointwise approximately unitarily equivalent to α.

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This seems to suggest that on $M_{|G|^\infty}$-absorbing C^*-algebras, there should be plenty of G-actions with the Rokhlin property, in particular with all kinds of K-theories.

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This seems to suggest that on $M_{|G|^\infty}$-absorbing C^*-algebras, there should be plenty of G-actions with the Rokhlin property, in particular with all kinds of K-theories.
However, it is in general not at all clear how many ordinary G-actions exist on a given C^*-algebra A, even if one assumes that A is classifiable.

Reminder

For a finite group action $\alpha : G \curvearrowright A$, the crossed product $A \rtimes_\alpha G$ is defined as the universal C^*-algebra generated by a copy of A, and a unitary representation $g \mapsto u_g$ subject to the relations $u_g a u_g^* = \alpha_g(a)$ for all $a \in A$.

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Reminder

Let us consider the special case $G = \mathbb{Z}_p$ for some $p \geq 2$. Set $\xi_p = \exp(2\pi i/p) \in \mathbb{C}$. Then a group action $\alpha : \mathbb{Z}_p \curvearrowright A$ naturally gives rise to the so-called dual action $\hat{\alpha} : \mathbb{Z}_p \curvearrowright A \rtimes_\alpha G$ by setting

$$\hat{\alpha}(u) = \xi_p u \quad \text{and} \quad \hat{\alpha}(a) = a \quad \text{for all } a \in A.$$

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One always has $(A \rtimes_\alpha \mathbb{Z}_p) \rtimes_{\hat{\alpha}} \mathbb{Z}_p \cong M_p \otimes A$.

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(All of this makes sense for actions of finite abelian groups as well.)

Definition

An action $\alpha : G \curvearrowright A$ is called locally representable, if there is an increasing sequence of unital, α-invariant sub-C^*-algebras $A_n \subset A$ with $A = \overline{\bigcup_{n\in\mathbb{N}} A_n}$, such that for all n, there is a unitary representation $w_n : G \to \mathcal{U}(A_n)$ such that $\alpha|_{A_n} = \mathrm{Ad}(w_n)$.

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Let $\mathfrak{C}$ be a class of C^*-algebras. α is called locally $\mathfrak{C}$-representable, if it is locally representable and the A_n above may be chosen to be isomorphic to C^*-algebras in $\mathfrak{C}$.

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Theorem (Barlak-S)

Assume $A \cong M_{|G|^\infty} \otimes A$. Let G be abelian and let $\alpha : G \curvearrowright A$ be a Rokhlin action. Then its dual $\hat{\alpha} : \hat{G} \curvearrowright A \rtimes_\alpha G$ is locally $\{A\}$-representable.

Notation

Let $\mathcal{R}_G(A)$ denote the set of all Rokhlin actions of G on A.

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The following will serve as the main black box for the rest of the talk:

Theorem (Barlak-S)

Let A be a unital UCT Kirchberg algebra. Assume that $A \cong M_{|G|^\infty} \otimes A$. Then the natural map

$$\mathcal{R}_G(A) \longrightarrow \mathrm{Hom}\Big(G, \mathrm{Aut}\big(K_0(A), [\mathbf{1}_A]_0, K_1(A)\big)\Big)$$

given by

$$[g \mapsto \alpha_g] \longmapsto [g \mapsto K_*(\alpha_g)]$$

is surjective.

Reminder (from Wilhelm's talk yesterday)

Let A, B be two unital Kirchberg algebras satisfying the UCT. Then

$$A \cong B \quad \text{iff} \quad \big(K_0(A), [\mathbf{1}_A]_0, K_1(A)\big) \cong \big(K_0(B), [\mathbf{1}_B]_0, K_1(B)\big)$$

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Moreover, any triple (G_0, u, G_1) for countable abelian groups $G_0 \ni u$ and G_1 arises as the K-theory triple of some unital UCT Kirchberg algebra.

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Moreover, any triple (G_0, u, G_1) for countable abelian groups $G_0 \ni u$ and G_1 arises as the K-theory triple of some unital UCT Kirchberg algebra.

Fact

Let $\alpha : G \curvearrowright A$ be a Rokhlin action.

- *If A is simple, so is $A \rtimes_\alpha G$.*
- *If A is purely infinite, so is $A \rtimes_\alpha G$.*
- *If A satisfies the UCT, so does $A \rtimes_\alpha G$.*

In particular, the class of (UCT) Kirchberg algebras is closed under forming crossed products by Rokhlin actions.

Fact

Let $p \geq 2$ *be a natural number. Let us pick a primitive* p*-th root of unity* $\xi_p = \exp(2\pi i/p) \in \mathbb{C}$*. Then the ring generated by* $\mathbb{Z}$ *and* ξ_p*, written* $\mathbb{Z}[\xi_p]$*, coincides with the ring of integers in the number field* $\mathbb{Q}(\xi_p)$*. The additive group of this ring is well-known to be free abelian, with rank equal to* $[\mathbb{Q}(\xi_p) : \mathbb{Q}]$*, which coincides with the value of Euler's phi-function at* p

$$\varphi(p) = \left| \{ j \in \{1, \dots, p\} \mid \gcd(j, p) = 1 \} \right|.$$

For example, if p *happens to be prime, then* $\varphi(p) = p - 1$*.*

Example

Let $p \geq 2$ be a natural number. Then there exists a locally UCT Kirchberg-representable action $\gamma_p : \mathbb{Z}_p \curvearrowright \mathcal{O}_2$ such that $A_p = \mathcal{O}_2 \rtimes_{\gamma_p} \mathbb{Z}_p$ is KK-equivalent to $M_{p^\infty}^{\oplus\varphi(p)}$.

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Proof: Choose a unital UCT Kirchberg algebra A_p with K-theory

$$(K_0(A_p), [\mathbf{1}_{A_p}]_0, K_1(A_p)) \cong (\mathbb{Z}[\tfrac{1}{p}]^{\oplus \varphi(p)}, 0, 0).$$

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$$(K_0(A_p), [\mathbf{1}_{A_p}]_0, K_1(A_p)) \cong (\mathbb{Z}[\tfrac{1}{p}]^{\oplus \varphi(p)}, 0, 0).$$

By the UCT, A_p is in fact KK-equivalent to $M_{p^\infty}^{\oplus \varphi(p)}$, since they have identical K-theory. Now $K_0(A_p)$ is isomorphic to the additive group of the ring $\mathbb{Z}[\frac{1}{p}, \xi_p]$. Under this identification, we obtain an order p automorphism $\sigma : K_0(A_p) \to K_0(A_p)$ by $x \mapsto \xi_p \cdot x$. Note that obviously $\ker(\mathrm{id} - \sigma) = 0$.

Since the K-theory of A_p is uniquely p-divisible, we have $A_p \cong M_{p^\infty} \otimes A_p$.

By our black box, there exists a Rokhlin action $\alpha : \mathbb{Z}_p \curvearrowright A_p$ with $K_0(\alpha) = \sigma$. Note that by the properties of σ, the crossed product $A_p \rtimes_\alpha \mathbb{Z}_p$ is a unital UCT Kirchberg algebra with trivial K-theory. Hence $A_p \rtimes_\alpha \mathbb{Z}_p \cong \mathcal{O}_2$.

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Under this identification, the dual action $\gamma_p = \hat{\alpha} : \mathbb{Z}_p \curvearrowright \mathcal{O}_2$ yields a locally $\{A_p\}$-representable action with

$$\mathcal{O}_2 \rtimes_{\gamma_p} \mathbb{Z}_p \cong (A_p \rtimes_\alpha \mathbb{Z}_p) \rtimes_{\hat{\alpha}} \mathbb{Z}_p \cong M_p \otimes A_p \cong A_p.$$

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But what do these actions have to do with the UCT problem?

Here are some well-known facts:

Fact

Let $n \in \mathbb{N}$ be a natural number and $A_1, \ldots, A_n$ separable C^*-algebras. Then each A_i satisfies the UCT if and only if $A_1 \oplus \cdots \oplus A_n$ satisfies the UCT.

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Fact

Let A be a separable C^*-algebra, and let $p, q \geq 2$ be two relatively prime natural numbers. Then A satisfies the UCT if and only if both $M_{p^\infty} \otimes A$ and $M_{q^\infty} \otimes A$ satisfy the UCT.

Here comes our main application concerning the UCT problem:

Theorem (partly Kirchberg, maybe even 'most' of it)

Let $p, q \geq 2$ be two distinct prime numbers. The following are equivalent:

(1) *All separable, nuclear C^*-algebras satisfy the UCT.*

(2) *All unital Kirchberg algebras satisfy the UCT.*

(3) *If $\beta : \mathbb{Z}_p \curvearrowright \mathcal{O}_2$ and $\gamma : \mathbb{Z}_q \curvearrowright \mathcal{O}_2$ are pointwise outer, locally Kirchberg-representable actions, then both $\mathcal{O}_2 \rtimes_\beta \mathbb{Z}_p$ and $\mathcal{O}_2 \rtimes_\gamma \mathbb{Z}_q$ satisfy the UCT.*

(4) *If $\gamma : \mathbb{Z}_{pq} \curvearrowright \mathcal{O}_2$ is a pointwise outer, locally Kirchberg-representable action, then $\mathcal{O}_2 \rtimes_\gamma \mathbb{Z}_{pq}$ satisfies the UCT.*

Proof: We will leave out anything involving (4).
The implications $(1) \implies (2)$ and $(2) \implies (3)$ are trivial. Let us show the implication $(3) \implies (2)$.

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Assume that (2) is false. Then we can pick a unital Kirchberg algebra A that does not satisfy the UCT. By the previous two facts, it follows that either $A \otimes M_{p^\infty}^{\oplus(p-1)}$ or $A \otimes M_{q^\infty}^{\oplus(q-1)}$ does not satisfy the UCT. Let us assume the first one.

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Recall the action $\gamma_p : \mathbb{Z}_p \curvearrowright \mathcal{O}_2$ from before. Then it follows that

$$(A \otimes \mathcal{O}_2) \rtimes_{\mathrm{id}_A \otimes \gamma_p} \mathbb{Z}_p \cong A \otimes A_p \sim_{KK} A \otimes M_{p^\infty}^{\oplus(p-1)}$$

does not satisfy the UCT. Recall that γ_p is pointwise outer and locally Kirchberg-representable. Moreover, Kirchberg's absorption theorem asserts $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$. In particular, this gives a counterexample to (3).

Lastly, let us sketch $(2) \implies (1)$, which is entirely due to Kirchberg.

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Definition

Let $p \in \mathcal{O}_\infty$ be some non-trivial projection with $0 = [p]_0 \in K_0(\mathcal{O}_\infty) = \mathbb{Z}$. Then define $\mathcal{O}_\infty^{\mathrm{st}} = p\mathcal{O}_\infty p$.

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Remark

Kirchberg-Phillips classification (in its more general form) tells us that $\mathcal{O}_\infty^{\mathrm{st}}$ is (up to isomorphism) the unique unital Kirchberg algebra with $\mathcal{O}_\infty^{\mathrm{st}} \sim_{KK} \mathbb{C}$ and which also admits a unital embedding of $\mathcal{O}_2$.

Now take any separable, nuclear C^*-algebra A. Out of pure convenience, we assume that A is unital. Without loss of generality, we may assume $A \cong A \otimes \mathcal{O}_\infty^{\mathrm{st}}$ by the previous remark. Since there is a unital embedding $\iota : \mathcal{O}_2 \to A$, pick $s_1, s_2 \in A$ with $\mathbf{1}_A = s_1^* s_1 = s_2^* s_2 = s_1 s_1^* + s_2 s_2^*$.

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There is also some unital embedding $\kappa : A \to \mathcal{O}_2$ by Kirchberg's embedding theorem. Define the unital endomorphism

$$\varphi : A \to A, \quad \varphi(x) = s_1 x s_1^* + s_2 (\iota \circ \kappa)(x) s_2^*.$$

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Set $B = \varinjlim \{A, \varphi\}$. Clearly B is again separable, unital, nuclear and purely infinite. One can also show quite easily that B is simple. Moreover, φ is KK-trivial. In such a case, the embedding $\varphi_\infty : A \to B$ is well-known to yield a KK-equivalence.

Now take any separable, nuclear C^*-algebra A. Out of pure convenience, we assume that A is unital. Without loss of generality, we may assume $A \cong A \otimes \mathcal{O}_\infty^{\mathrm{st}}$ by the previous remark. Since there is a unital embedding $\iota : \mathcal{O}_2 \to A$, pick $s_1, s_2 \in A$ with $\mathbf{1}_A = s_1^* s_1 = s_2^* s_2 = s_1 s_1^* + s_2 s_2^*$.

There is also some unital embedding $\kappa : A \to \mathcal{O}_2$ by Kirchberg's embedding theorem. Define the unital endomorphism

$$\varphi : A \to A, \quad \varphi(x) = s_1 x s_1^* + s_2 (\iota \circ \kappa)(x) s_2^*.$$

Set $B = \varinjlim \{A, \varphi\}$. Clearly B is again separable, unital, nuclear and purely infinite. One can also show quite easily that B is simple. Moreover, φ is KK-trivial. In such a case, the embedding $\varphi_\infty : A \to B$ is well-known to yield a KK-equivalence.

To summarize, we have found a unital Kirchberg algebra that is KK-equivalent to A. This yields the implication $(1) \implies (2)$.

Thank you for your attention!