Sequentially split $*$-homomorphisms (Part II)

Workshop on Structure and Classification of C^*-algebras

Selçuk Barlak
(joint with Gábor Szabó)

WWU Münster

April 2015

A word of warning: This talk describes work in progress, and the proofs of the results still need to be checked in detail. Do not quote them yet!

1. Equivariantly sequentially split $*$-homomorphisms
2. Rokhlin actions
3. Extending Izumi's duality result

1. Equivariantly sequentially split $*$-homomorphisms
2. Rokhlin actions
3. Extending Izumi's duality result

In this talk, all groups are supposed to be second countable and locally compact.

In this talk, all groups are supposed to be second countable and locally compact.

Definition

Let A be a C^*-algebra, G a group, $\alpha : G \curvearrowright A$ a (point-norm) continuous action and ω a free filter on $\mathbb{N}$.

In this talk, all groups are supposed to be second countable and locally compact.

Definition

Let A be a C^*-algebra, G a group, $\alpha : G \curvearrowright A$ a (point-norm) continuous action and ω a free filter on $\mathbb{N}$. Componentwise application of α yields a (possibly not continuous) action $\alpha_\omega : G \curvearrowright A_\omega$.

In this talk, all groups are supposed to be second countable and locally compact.

Definition

Let A be a C^*-algebra, G a group, $\alpha : G \curvearrowright A$ a (point-norm) continuous action and ω a free filter on $\mathbb{N}$. Componentwise application of α yields a (possibly not continuous) action $\alpha_\omega : G \curvearrowright A_\omega$. We define

$$A_{\omega,\alpha} = \{x \in A_\omega \mid [g \mapsto \alpha_{\omega,g}(x)] \text{ is continuous}\} .$$

This yields a continuous action $\alpha_\omega : G \curvearrowright A_{\omega,\alpha}$.

In this talk, all groups are supposed to be second countable and locally compact.

Definition

Let A be a C^*-algebra, G a group, $\alpha : G \curvearrowright A$ a (point-norm) continuous action and ω a free filter on $\mathbb{N}$. Componentwise application of α yields a (possibly not continuous) action $\alpha_\omega : G \curvearrowright A_\omega$. We define

$$A_{\omega,\alpha} = \{x \in A_\omega \mid [g \mapsto \alpha_{\omega,g}(x)] \text{ is continuous}\}\,.$$

This yields a continuous action $\alpha_\omega : G \curvearrowright A_{\omega,\alpha}$.

Definition

Let A and B be C^*-algebras, G a group and $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ continuous actions.

In this talk, all groups are supposed to be second countable and locally compact.

Definition

Let A be a C^*-algebra, G a group, $\alpha : G \curvearrowright A$ a (point-norm) continuous action and ω a free filter on $\mathbb{N}$. Componentwise application of α yields a (possibly not continuous) action $\alpha_\omega : G \curvearrowright A_\omega$. We define

$$A_{\omega,\alpha} = \{x \in A_\omega \mid [g \mapsto \alpha_{\omega,g}(x)] \text{ is continuous}\}.$$

This yields a continuous action $\alpha_\omega : G \curvearrowright A_{\omega,\alpha}$.

Definition

Let A and B be C^*-algebras, G a group and $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ continuous actions. An equivariant $*$-homomorphism $\varphi : (A,\alpha) \to (B,\beta)$ is called (equivariantly) sequentially split, if there exists an equivariant $*$-homomorphism $\psi : (B,\beta) \to (A_{\infty,\alpha}, \alpha_\infty)$ such that the composition $\psi \circ \varphi$ coincides with the standard embedding of A into $A_{\infty,\alpha}$.

Definition (continued)

In other words, there exists a commutative diagram

$$\begin{array}{ccc} (A,\alpha) & \xrightarrow{\hspace{3em}} & (A_{\infty,\alpha},\alpha_\infty) \\ & {\scriptstyle\varphi}\searrow \quad \nearrow & \\ & (B,\beta) & \end{array}$$

of equivariant $*$-homomorphisms.

Definition (continued)

In other words, there exists a commutative diagram

$$\begin{array}{ccc} (A,\alpha) & \longrightarrow & (A_{\infty,\alpha},\alpha_{\infty}) \\ & {\scriptstyle\varphi}\searrow \qquad \nearrow & \\ & (B,\beta) & \end{array}$$

of equivariant $*$-homomorphisms.

Remark

If one restricts to separable C^*-algebras, one gets an equivalent definition upon replacing $(A_{\infty,\alpha},\alpha_{\infty})$ by $(A_{\omega,\alpha},\alpha_{\omega})$, for any free filter ω on $\mathbb{N}$.

Like its non-equivariant counterpart, this notion is well-behaved under some standard constructions.

Proposition

If the involved C^*-algebras are separable, then the composition of two equivariantly sequentially split $*$-homomorphisms is equivariantly sequentially split.

Like its non-equivariant counterpart, this notion is well-behaved under some standard constructions.

Proposition

If the involved C^*-algebras are separable, then the composition of two equivariantly sequentially split $*$-homomorphisms is equivariantly sequentially split.

Proposition

Let $\varphi : (A, \alpha) \to (B, \beta)$ and $\psi : (C, \gamma) \to (D, \delta)$ be two sequentially split $*$-homomorphisms. Then

$$\varphi \otimes \psi : (A \otimes_{\max} B, \alpha \otimes \beta) \to (C \otimes_{\max} D, \gamma \otimes \delta)$$

is sequentially split.

Like its non-equivariant counterpart, this notion is well-behaved under some standard constructions.

Proposition

If the involved C^*-algebras are separable, then the composition of two equivariantly sequentially split $*$-homomorphisms is equivariantly sequentially split.

Proposition

Let $\varphi : (A, \alpha) \to (B, \beta)$ and $\psi : (C, \gamma) \to (D, \delta)$ be two sequentially split $*$-homomorphisms. Then

$$\varphi \otimes \psi : (A \otimes_{\max} B, \alpha \otimes \beta) \to (C \otimes_{\max} D, \gamma \otimes \delta)$$

is sequentially split.

In analogy to the non-equivariant case, equivariantly sequentially split $*$-homomorphisms are also well-behaved with respect to equivariant inductive limits.

Proposition

Let A and B be C^*-algebras, G a group and $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ continuous actions. Assume that $\varphi : (A, \alpha) \to (B, \beta)$ is a sequentially split $*$-homomorphism. Then:

- The induced $*$-homomorphism $\varphi \rtimes G : A \rtimes_\alpha G \to B \rtimes_\beta G$ between the crossed products is sequentially split.

Proposition

Let A and B be C^*-algebras, G a group and $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ continuous actions. Assume that $\varphi : (A, \alpha) \to (B, \beta)$ is a sequentially split $*$-homomorphism. Then:

- The induced $*$-homomorphism $\varphi \rtimes G : A \rtimes_\alpha G \to B \rtimes_\beta G$ between the crossed products is sequentially split.
- If G is compact, then the induced $*$-homomorphism $\varphi : A^\alpha \to B^\beta$ between the fixed point algebras is sequentially split.

Proposition

Let A and B be C^*-algebras, G a group and $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ continuous actions. Assume that $\varphi : (A, \alpha) \to (B, \beta)$ is a sequentially split $*$-homomorphism. Then:

- The induced $*$-homomorphism $\varphi \rtimes G : A \rtimes_\alpha G \to B \rtimes_\beta G$ between the crossed products is sequentially split.
- If G is compact, then the induced $*$-homomorphism $\varphi : A^\alpha \to B^\beta$ between the fixed point algebras is sequentially split.

One has the following Takai Duality-type result:

Proposition

Let A and B be C^*-algebras, G a group and $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ continuous actions. Assume that $\varphi : (A, \alpha) \to (B, \beta)$ is a sequentially split $*$-homomorphism. Then:

- The induced $*$-homomorphism $\varphi \rtimes G : A \rtimes_\alpha G \to B \rtimes_\beta G$ between the crossed products is sequentially split.
- If G is compact, then the induced $*$-homomorphism $\varphi : A^\alpha \to B^\beta$ between the fixed point algebras is sequentially split.

One has the following Takai Duality-type result:

Theorem

Let A and B be σ-unital C^*-algebras, G an abelian group and $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ continuous actions. An equivariant $*$-homomorphism $\varphi : (A, \alpha) \to (B, \beta)$ is sequentially split **if and only if** *the dual morphism $\hat{\varphi} : (A \rtimes_\alpha G, \hat{\alpha}) \to (B \rtimes_\beta G, \hat{\beta})$ is ($\hat{G}$-equivariantly) sequentially split.*

Corollary

Let A and B be separable C^*-algebras and let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be continuous actions. Assume that $\varphi : (A, \alpha) \to (B, \beta)$ is a non-degenerate, sequentially split $*$-homomorphism.

Corollary

Let A and B be separable C^*-algebras and let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be continuous actions. Assume that $\varphi : (A, \alpha) \to (B, \beta)$ is a non-degenerate, sequentially split $*$-homomorphism. Then all the properties listed in the last talk pass from $B \rtimes_\beta G$ to $A \rtimes_\alpha G$.

Corollary

Let A and B be separable C^*-algebras and let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be continuous actions. Assume that $\varphi : (A, \alpha) \to (B, \beta)$ is a non-degenerate, sequentially split $*$-homomorphism. Then all the properties listed in the last talk pass from $B \rtimes_\beta G$ to $A \rtimes_\alpha G$. If G is compact, then the same is true for the fixed point algebras B^β and A^α.

1. Equivariantly sequentially split $*$-homomorphisms

2. Rokhlin actions

3. Extending Izumi's duality result

Definition (following Kirchberg '04)

Let A be a C^*-algebra. The central sequence algebra of A is defined as the quotient

$$F_\infty(A) = A_\infty \cap A' / \{x \in A_\infty \mid xA + Ax = 0\}.$$

Definition (following Kirchberg '04)

Let A be a C^*-algebra. The central sequence algebra of A is defined as the quotient

$$F_\infty(A) = A_\infty \cap A' / \{x \in A_\infty \mid xA + Ax = 0\}.$$

Let G be a group and $\alpha : G \curvearrowright A$ a continuous action. α induces a (possibly not continuous) G-action α_∞ on $F_\infty(A)$.

Definition (following Kirchberg '04)

Let A be a C^*-algebra. The central sequence algebra of A is defined as the quotient

$$F_\infty(A) = A_\infty \cap A' / \{x \in A_\infty \mid xA + Ax = 0\}.$$

Let G be a group and $\alpha : G \curvearrowright A$ a continuous action. α induces a (possibly not continuous) G-action α_∞ on $F_\infty(A)$. We define

$$F_{\infty,\alpha}(A) = \{x \in F_\infty(A) \mid [g \mapsto \alpha_{\infty,g}(x)] \text{ is continuous}\}.$$

This yields a continuous action $\alpha_\infty : G \curvearrowright F_{\infty,\alpha}(A)$.

Definition (following Kirchberg '04)

Let A be a C^*-algebra. The central sequence algebra of A is defined as the quotient

$$F_\infty(A) = A_\infty \cap A' / \{x \in A_\infty \mid xA + Ax = 0\}.$$

Let G be a group and $\alpha : G \curvearrowright A$ a continuous action. α induces a (possibly not continuous) G-action α_∞ on $F_\infty(A)$. We define

$$F_{\infty,\alpha}(A) = \{x \in F_\infty(A) \mid [g \mapsto \alpha_{\infty,g}(x)] \text{ is continuous}\}.$$

This yields a continuous action $\alpha_\infty : G \curvearrowright F_{\infty,\alpha}(A)$.

Remark

If A is σ-unital, then $F_\infty(A)$ is unital. In this case, the unit is represented by any countable approximate unit for A.

Definition

Let A be a separable C^*-algebra and G a compact group. A continuous action $\alpha : G \curvearrowright A$ is said to have the Rokhlin property, if there exists a unital and equivariant $*$-homomorphism

$$(\mathcal{C}(G), \sigma) \to (F_{\infty,\alpha}(A), \alpha_\infty),$$

where σ denotes the canonical G-shift.

Definition

Let A be a separable C^*-algebra and G a compact group. A continuous action $\alpha : G \curvearrowright A$ is said to have the Rokhlin property, if there exists a unital and equivariant $*$-homomorphism

$$(\mathcal{C}(G), \sigma) \to (F_{\infty,\alpha}(A), \alpha_\infty),$$

where σ denotes the canonical G-shift.

We can characterize actions with the Rokhlin property in terms of equivariantly sequentially split $*$-homomorphisms:

Definition

Let A be a separable C^*-algebra and G a compact group. A continuous action $\alpha : G \curvearrowright A$ is said to have the Rokhlin property, if there exists a unital and equivariant $*$-homomorphism

$$(\mathcal{C}(G), \sigma) \to (F_{\infty,\alpha}(A), \alpha_\infty),$$

where σ denotes the canonical G-shift.

We can characterize actions with the Rokhlin property in terms of equivariantly sequentially split $*$-homomorphisms:

Proposition

Let A be a separable C^*-algebra, G a compact group and $\alpha : G \curvearrowright A$ a continuous action. Then α has the Rokhlin property if and only if

$$\mathrm{id}_A \otimes \mathbf{1} : (A, \alpha) \longrightarrow (A \otimes \mathcal{C}(G), \alpha \otimes \sigma)$$

is sequentially split.

Theorem

Let A *be a separable* C^**-algebra,* G *a compact group and* $\alpha : G \curvearrowright A$ *an action with the Rokhlin property. Then the natural inclusions* $A^\alpha \hookrightarrow A$ *and* $A \rtimes_\alpha G \hookrightarrow A \otimes \mathcal{K}(L^2(G))$ *are sequentially split.*

Theorem

Let A *be a separable* C^**-algebra,* G *a compact group and* $\alpha : G \curvearrowright A$ *an action with the Rokhlin property. Then the natural inclusions* $A^\alpha \hookrightarrow A$ *and* $A \rtimes_\alpha G \hookrightarrow A \otimes \mathcal{K}(L^2(G))$ *are sequentially split. In particular, all the properties listed in the last talk pass from* A *to* A^α *and from* $A \otimes \mathcal{K}(L^2(G))$ *to* $A \rtimes_\alpha G$*.*

Theorem

Let A be a separable C^*-algebra, G a compact group and $\alpha : G \curvearrowright A$ an action with the Rokhlin property. Then the natural inclusions $A^\alpha \hookrightarrow A$ and $A \rtimes_\alpha G \hookrightarrow A \otimes \mathcal{K}(L^2(G))$ are sequentially split. In particular, all the properties listed in the last talk pass from A to A^α and from $A \otimes \mathcal{K}(L^2(G))$ to $A \rtimes_\alpha G$.

Corollary

Let A be a separable, nuclear C^*-algebra, G a compact group and $\alpha : G \curvearrowright A$ an action with the Rokhlin property. If A satisfies the UCT, then A^α and $A \rtimes_\alpha G$ also satisfy the UCT.

1 Equivariantly sequentially split $*$-homomorphisms

2 Rokhlin actions

3 Extending Izumi's duality result

Definition

Let A be a separable C^*-algebra and H a discrete group. An action $\alpha : H \curvearrowright A$ is called approximately representable, if there exist contractions $x_{n,h} \in A$, $n \in \mathbb{N}$ and $h \in H$, satisfying the following relations:

Definition

Let A be a separable C^*-algebra and H a discrete group. An action $\alpha : H \curvearrowright A$ is called approximately representable, if there exist contractions $x_{n,h} \in A$, $n \in \mathbb{N}$ and $h \in H$, satisfying the following relations:

(1) For $h \in H$, $(x_{n,h}x_{n,h}^*)_n$ and $(x_{n,h}^*x_{n,h})_n$ are approximate units for A.

Definition

Let A be a separable C^*-algebra and H a discrete group. An action $\alpha : H \curvearrowright A$ is called approximately representable, if there exist contractions $x_{n,h} \in A$, $n \in \mathbb{N}$ and $h \in H$, satisfying the following relations:

(1) For $h \in H$, $(x_{n,h}x_{n,h}^*)_n$ and $(x_{n,h}^*x_{n,h})_n$ are approximate units for A.

(2) For all $g, h \in H$ and $a \in A$,

$$\lim_{n\to\infty} \|a(x_{n,g}x_{n,h} - x_{n,gh})\| + \|(x_{n,g}x_{n,h} - x_{n,gh})a\| = 0.$$

Definition

Let A be a separable C^*-algebra and H a discrete group. An action $\alpha : H \curvearrowright A$ is called approximately representable, if there exist contractions $x_{n,h} \in A$, $n \in \mathbb{N}$ and $h \in H$, satisfying the following relations:

(1) For $h \in H$, $(x_{n,h}x_{n,h}^*)_n$ and $(x_{n,h}^*x_{n,h})_n$ are approximate units for A.

(2) For all $g, h \in H$ and $a \in A$,

$$\lim_{n\to\infty} \|a(x_{n,g}x_{n,h} - x_{n,gh})\| + \|(x_{n,g}x_{n,h} - x_{n,gh})a\| = 0.$$

(3) For all $a \in A$ and $h \in H$, $\lim_{n\to\infty} \|\alpha_h(a) - x_{n,h}ax_{n,h}^*\| = 0$,

Definition

Let A be a separable C^*-algebra and H a discrete group. An action $\alpha : H \curvearrowright A$ is called approximately representable, if there exist contractions $x_{n,h} \in A$, $n \in \mathbb{N}$ and $h \in H$, satisfying the following relations:

(1) For $h \in H$, $(x_{n,h}x_{n,h}^*)_n$ and $(x_{n,h}^*x_{n,h})_n$ are approximate units for A.

(2) For all $g, h \in H$ and $a \in A$,

$$\lim_{n\to\infty} \|a(x_{n,g}x_{n,h} - x_{n,gh})\| + \|(x_{n,g}x_{n,h} - x_{n,gh})a\| = 0.$$

(3) For all $a \in A$ and $h \in H$, $\lim_{n\to\infty} \|\alpha_h(a) - x_{n,h}ax_{n,h}^*\| = 0$,

(4) For all $g, h \in H$ and $a \in A$.

$$\lim_{n\to\infty} \|a(x_{n,ghg^{-1}} - \alpha_g(x_{n,h}))\| + \|(x_{n,ghg^{-1}} - \alpha_g(x_{n,h}))a\| = 0.$$

Definition

Let A be a separable C^*-algebra and H a discrete group. An action $\alpha : H \curvearrowright A$ is called approximately representable, if there exist contractions $x_{n,h} \in A$, $n \in \mathbb{N}$ and $h \in H$, satisfying the following relations:

(1) For $h \in H$, $(x_{n,h}x_{n,h}^*)_n$ and $(x_{n,h}^*x_{n,h})_n$ are approximate units for A.

(2) For all $g, h \in H$ and $a \in A$,

$$\lim_{n\to\infty} \|a(x_{n,g}x_{n,h} - x_{n,gh})\| + \|(x_{n,g}x_{n,h} - x_{n,gh})a\| = 0.$$

(3) For all $a \in A$ and $h \in H$, $\lim_{n\to\infty} \|\alpha_h(a) - x_{n,h}ax_{n,h}^*\| = 0$,

(4) For all $g, h \in H$ and $a \in A$.

$$\lim_{n\to\infty} \|a(x_{n,ghg^{-1}} - \alpha_g(x_{n,h}))\| + \|(x_{n,ghg^{-1}} - \alpha_g(x_{n,h}))a\| = 0.$$

Remark

In the unital case, one recovers the usual definition by Izumi.

Like the Rokhlin property, approximate representability can be characterized in terms of sequentially split $*$-homomorphisms.

Like the Rokhlin property, approximate representability can be characterized in terms of sequentially split $*$-homomorphisms.

Proposition

Let A be a separable C^*-algebra, H a discrete group and $\alpha : H \curvearrowright A$ an action. Then α is approximately representable if and only if

$$\iota_A : (A, \alpha) \hookrightarrow (A \rtimes_\alpha H, \mathrm{Ad}(\lambda^\alpha))$$

is sequentially split.

Here $\lambda^\alpha : H \to \mathcal{U}(\mathcal{M}(A \rtimes_\alpha H))$ denotes the canonical unitary represention implementing α.

Recall Izumi's duality result concerning the Rokhlin property and approximate representability for finite abelian group actions on unital, separable C^*-algebras:

Recall Izumi's duality result concerning the Rokhlin property and approximate representability for finite abelian group actions on unital, separable C^*-algebras:

Theorem (Izumi '04)

Let A be a unital, separable C^*-algebra, G a finite abelian group and $\alpha : G \curvearrowright A$ an action. Then

(1) *α has the Rokhlin property if and only if $\hat{\alpha}$ is approximately representable,*

Recall Izumi's duality result concerning the Rokhlin property and approximate representability for finite abelian group actions on unital, separable C^*-algebras:

Theorem (Izumi '04)

Let A be a unital, separable C^*-algebra, G a finite abelian group and $\alpha : G \curvearrowright A$ an action. Then

(1) *α has the Rokhlin property if and only if $\hat{\alpha}$ is approximately representable,*

(2) *α is approximately representable if and only if $\hat{\alpha}$ has the Rokhlin property.*

Using the above characterization of approximate representability and the Takai Duality-type result for equivariantly sequentially split $*$-homomorphisms, we can extend Izumi's result as follows.

Using the above characterization of approximate representability and the Takai Duality-type result for equivariantly sequentially split $*$-homomorphisms, we can extend Izumi's result as follows.

Theorem

Let A *be a separable* C^**-algebra,* G *a compact, abelian group and* H *a discrete, abelian group. Let* $\alpha : G \curvearrowright A$ *and* $\beta : H \curvearrowright A$ *be two continuous actions. Then*

(1) α *has the Rokhlin property if and only if* $\hat{\alpha}$ *is approximately representable,*

Using the above characterization of approximate representability and the Takai Duality-type result for equivariantly sequentially split $*$-homomorphisms, we can extend Izumi's result as follows.

Theorem

Let A be a separable C^*-algebra, G a compact, abelian group and H a discrete, abelian group. Let $\alpha : G \curvearrowright A$ and $\beta : H \curvearrowright A$ be two continuous actions. Then

(1) *α has the Rokhlin property if and only if $\hat{\alpha}$ is approximately representable,*

(2) *β is approximately representable if and only if $\hat{\beta}$ has the Rokhlin property.*

Thank you for your attention!