On the nuclear dimension of strongly purely infinite C^*-algebras

Workshop on Noncommutative Dimension Theories, Honolulu

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1. Nuclear dimension and $\mathcal{Z}$-stability

2. Strongly purely infinite C^*-algebras

3. A dimension reduction argument

4. On Rørdam's purely infinite AH algebra

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For a non-elementary, separable, nuclear, simple, unital C^*-algebra A, TFAE:

(1) $\dim_{\mathrm{nuc}}(A) < \infty$;

(2) $A \cong A \otimes \mathcal{Z}$;

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In this talk, we shall mainly be focused on "$(1) \iff (2)$". The implication "$(1) \implies (2)$" is due to Winter and is very non-trivial. The implication "$(2) \implies (1)$" is more mysterious, but has seen progress lately.

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In particular:

Question

Is $\dim_{\mathrm{nuc}}(A \otimes \mathcal{Z}) < \infty$ for every separable, nuclear C^*-algebra A?

Some results in the direction of this general conjecture:

Theorem (Robert-Tikuisis)

Let A be a separable, nuclear C^*-algebra without elementary quotients. Assume that no simple quotient of A is purely infinite, and that $\mathrm{Prim}(A)$ is either Hausdorff or has a basis of compact-open sets. If $\dim_{\mathrm{nuc}}(A) < \infty$, then $A \cong A \otimes \mathcal{Z}$.

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Theorem (Tikuisis-Winter)

One has $\mathrm{dr}(\mathcal{C}_0(X) \otimes \mathcal{Z}) \leq 2$ for every locally compact space X.

Definition (Kirchberg-Rørdam)

A C^*-algebra A is called strongly purely infinite, if for every positive matrix $\begin{pmatrix} a_1 & x^* \\ x & a_2 \end{pmatrix} \in M_2(A)$ and $\varepsilon > 0$, there exist $d_1, d_2 \in A$ satisfying

$$\left\| \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}^* \begin{pmatrix} a_1 & x^* \\ x & a_2 \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} - \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \right\| \leq \varepsilon.$$

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Remark

If A is simple, this coincides with the *usual* definition of pure infiniteness.

Theorem (Kirchberg-Rørdam, Toms-Winter, Kirchberg)

Let A be a separable, nuclear C^*-algebra. TFAE:

(1) *A is strongly purely infinite;*

(2) $A \cong A \otimes \mathcal{O}_\infty$;

(3) $A \cong A \otimes \mathcal{Z}$ *and A is traceless.*

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Today I would like to convince you that: **Yes!**

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Theorem (Kirchberg)

Let A and B be two separable, nuclear, stable, strongly purely infinite C^*-algebras. Then $A \cong B$, if and only if, $X := \mathrm{Prim}(A) \cong \mathrm{Prim}(B)$ and there exists a $KK(X; \_, \_)$-equivalence from A to B.

Let us first recall a well-known special case of our main question:

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Sketch of proof for the dimension formula.

Find pairs of c.p.c. $\approx$-order zero maps $\varphi_0, \varphi_1 : \mathcal{O}_2 \to \mathcal{O}_\infty$ with $\varphi_0(\mathbf{1}) + \varphi_1(\mathbf{1}) \approx \mathbf{1}$, and use the fact that $\mathcal{O}_\infty$ is strongly self-absorbing. □

Let us now consider a more general *2-colored embedding result*:

Theorem (S)

Let ω be a free ultrafilter. Let A be a separable C^*-algebra and $e \in A$ a positive element of norm one. Then there exist two c.p.c. order zero maps $\varphi_0, \varphi_1 : A \to (\mathcal{O}_\infty)_\omega$ with $\varphi_0(e) + \varphi_1(e) = \mathbf{1}$.

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For this we need an observation from BEMSW:

Lemma (Winter)

In a unital, simple, purely infinite C^*-algebra, all positive elements with full spectrum $[0, 1]$ are mutually approximately unitarily equivalent.

Proof.

Since $\mathcal{O}_\infty$ contains the compact operators $\mathcal{K}$, every separable, quasidiagonal C^*-algebra embeds into $(\mathcal{O}_\infty)_\omega$. By a result of Voiculescu, the cone over A is quasidiagonal. So we can find a $*$-monomorphism $\psi : CA \to (\mathcal{O}_\infty)_\omega$.

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Theorem (S)

Let B be a separable, non-zero, $\mathcal{O}_\infty$-absorbing C^*-algebra. Then for every separable C^*-algebra A, we have $\dim^{+1}_{\mathrm{nuc}}(A \otimes \mathcal{O}_\infty) \leq 2\dim^{+1}_{\mathrm{nuc}}(A \otimes B)$.

Theorem (continued)

Let B be a separable, non-zero, $\mathcal{O}_\infty$-absorbing C^*-algebra. Then for every separable C^*-algebra A, we have $\dim_{\mathrm{nuc}}^{+1}(A \otimes \mathcal{O}_\infty) \leq 2 \dim_{\mathrm{nuc}}^{+1}(A \otimes B)$.

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Proof.

We may assume that $A \cong A \otimes \mathcal{O}_\infty$. As B is non-zero, we may choose some positive element $e \in B$ of norm one. Apply the 2-colored embedding theorem to find c.p.c. order zero maps $\varphi_0, \varphi_1 : B \to (\mathcal{O}_\infty)_\omega$ with $\varphi_0(e) + \varphi_1(e) = \mathbf{1}$.

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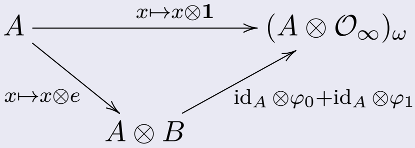

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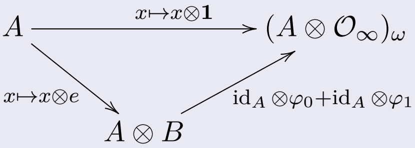

Since $A \cong A \otimes \mathcal{O}_\infty$, the nuclear dimension of the horizontal map equals the nuclear dimension of A. This shows the claim. □

Definition

Let $\{t_n\}_{n\in\mathbb{N}} \subset [0,1)$ be a dense sequence. For every n, define the $*$-homomorphism

$$\varphi_n : \mathcal{C}_0\big([0,1), M_{2^n}\big) \to \mathcal{C}_0\big([0,1), M_{2^{n+1}}\big)$$

via

$$\varphi_n(f)(t) = \operatorname{diag}\Big(f(t), f\big(\max(t,t_n)\big)\Big) \quad \text{for all } t \in [0,1).$$

Set $\mathcal{A}_{[0,1]} = \varinjlim \{\mathcal{C}([0,1), M_{2^n}), \varphi_n\}$.

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Theorem (Rørdam)

$\mathcal{A}_{[0,1]}$ is $\mathcal{O}_\infty$-absorbing.

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A C^*-algebra A is homotopic to zero in an ideal-system preserving way, if there is a continuous path of $*$-endomorphisms $\{\rho_t\}_{t\in[0,1]}$ with $\rho_0 = 0$, $\rho_1 = \mathrm{id}_A$ and $\rho_t(J) \subset J$ for every $t \in [0,1]$ and all ideals $J \subset A$.

The class of such nuclear C^*-algebras is closed under tensoring with arbitrary separable, nuclear C^*-algebras.

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Theorem (Kirchberg-Rørdam)

Let A be a separable, nuclear, strongly purely infinite C^*-algebra that is homotopic to zero in an ideal-system preserving way. Then A is an AH algebra of topological dimension one.

By combining this deep structural result with the previous dimension reduction argument, we get:

Theorem

For every separable, nuclear C^**-algebra* A*, we have* $\dim_{\mathrm{nuc}}(A \otimes \mathcal{O}_\infty) \leq 3$.

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Proof.

Assume $A \neq 0$. By the results of Kirchberg-Rørdam, the tensor product $A \otimes \mathcal{A}_{[0,1]}$ is an AH algebra with topological dimension one. Thus

$$\dim_{\mathrm{nuc}}^{+1}(A \otimes \mathcal{O}_\infty) \leq 2 \dim_{\mathrm{nuc}}^{+1}(A \otimes \mathcal{A}_{[0,1]}) = 4.$$

This gives $\dim_{\mathrm{nuc}}(A \otimes \mathcal{O}_\infty) \leq 3$. □

Thank you for your attention!