Strongly self-absorbing C^*-dynamical systems

Classification and dynamical systems I: C^*-algebras
Mittag-Leffler institute, Stockholm

Gábor Szabó

WWU Münster

February 2016

At various points in time, significant advances in the Elliott program have shown that a class of C^*-algebras can be reasonably handled from a classification point of view, if one assumes certain regularity properties for the C^*-algebras in this class. This ties into the Toms-Winter conjecture.

At various points in time, significant advances in the Elliott program have shown that a class of C^*-algebras can be reasonably handled from a classification point of view, if one assumes certain regularity properties for the C^*-algebras in this class. This ties into the Toms-Winter conjecture.

One of these regularity properties concerns the tensorial absorption of some strongly self-absorbing C^*-algebra $\mathcal{D}$.

At various points in time, significant advances in the Elliott program have shown that a class of C^*-algebras can be reasonably handled from a classification point of view, if one assumes certain regularity properties for the C^*-algebras in this class. This ties into the Toms-Winter conjecture.

One of these regularity properties concerns the tensorial absorption of some strongly self-absorbing C^*-algebra $\mathcal{D}$.

Already in Kirchberg-Phillips' classification of purely infinite C^*-algebras, the Cuntz algebra $\mathcal{O}_\infty$ played this role. Together with $\mathcal{O}_2$, these two objects are the cornerstones of that classification.

At various points in time, significant advances in the Elliott program have shown that a class of C^*-algebras can be reasonably handled from a classification point of view, if one assumes certain regularity properties for the C^*-algebras in this class. This ties into the Toms-Winter conjecture.

One of these regularity properties concerns the tensorial absorption of some strongly self-absorbing C^*-algebra $\mathcal{D}$.

Already in Kirchberg-Phillips' classification of purely infinite C^*-algebras, the Cuntz algebra $\mathcal{O}_\infty$ played this role. Together with $\mathcal{O}_2$, these two objects are the cornerstones of that classification.

In a very influential paper, the term of '*localizing the Elliott conjecture at a strongly self-absorbing* C^**-algebra* $\mathcal{D}$' was coined by Winter. The most general case concerns $\mathcal{D} = \mathcal{Z}$.

With the unital Elliott classification program approaching its conclusion, it can be inspiring to have a look at a fascinating string of results in the theory of von Neumann algebras, which initially paralleled and then followed the classification of injective factors:

With the unital Elliott classification program approaching its conclusion, it can be inspiring to have a look at a fascinating string of results in the theory of von Neumann algebras, which initially paralleled and then followed the classification of injective factors:

Theorem (Connes, Jones, Ocneanu, Sutherland-Takesaki, Kawahigashi-Sutherland-Takesaki, Katayama-Sutherland-Takesaki)

Let M be an injective factor and G a discrete amenable group. Then two pointwise outer G-actions on M are cocycle conjugugate by an approximately inner automorphism if and only if they agree on the Connes-Takesaki module.

With the unital Elliott classification program approaching its conclusion, it can be inspiring to have a look at a fascinating string of results in the theory of von Neumann algebras, which initially paralleled and then followed the classification of injective factors:

Theorem (Connes, Jones, Ocneanu, Sutherland-Takesaki, Kawahigashi-Sutherland-Takesaki, Katayama-Sutherland-Takesaki)

Let M be an injective factor and G a discrete amenable group. Then two pointwise outer G-actions on M are cocycle conjugugate by an approximately inner automorphism if and only if they agree on the Connes-Takesaki module.

More recently, Masuda has found a unified approach for McDuff-factors based on Evans-Kishimoto intertwining. Moreover, there exist also many convincing results of this spirit beyond the discrete group case.

Question

Can we classify C^*-dynamical systems?

Question

Can we classify C^*-dynamical systems?

In general, this is completely out of reach. In contrast to von Neumann algebras, there are major obstructions coming from K-theory.

Question

Can we classify C^*-dynamical systems?

In general, this is completely out of reach. In contrast to von Neumann algebras, there are major obstructions coming from K-theory.

Nevertheless, many people have invented novel approaches to make progress on this question.

Question

Can we classify C^*-dynamical systems?

In general, this is completely out of reach. In contrast to von Neumann algebras, there are major obstructions coming from K-theory.

Nevertheless, many people have invented novel approaches to make progress on this question. A bit of name-dropping: Herman, Jones, Ocneanu, Evans, Kishimoto, Elliott, Bratteli, Robinson, Izumi, Phillips, Nakamura, Lin, Katsura, Gardella, Santiago, **Matui**, **Sato**...**(impressive!)**

Question

Can we classify C^*-dynamical systems?

In general, this is completely out of reach. In contrast to von Neumann algebras, there are major obstructions coming from K-theory.

Nevertheless, many people have invented novel approaches to make progress on this question. A bit of name-dropping: Herman, Jones, Ocneanu, Evans, Kishimoto, Elliott, Bratteli, Robinson, Izumi, Phillips, Nakamura, Lin, Katsura, Gardella, Santiago, **Matui**, **Sato**...**(impressive!)**

Motivated by the importance of strongly self-absorbing C^*-algebras in the Elliott program, we ask:

Question

- Is there a dynamical analogue of a strongly self-absorbing C^*-algebra?
- Can we classify C^*-dynamical systems that absorb such objects?

From now, let G always denote a second-countable, locally compact group.

Definition

Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ denote actions on separable, unital C*-algebras. Let $\varphi_1, \varphi_2 : (A, \alpha) \to (B, \beta)$ be two equivariant and unital $*$-homomorphisms. We say that φ_1 and φ_2 are approximately G-unitarily equivalent, denoted $\varphi_1 \approx_{\mathrm{u},G} \varphi_2$, if there is a sequence of unitaries $v_n \in B$ with

$$\mathrm{Ad}(v_n) \circ \varphi_1 \stackrel{n\to\infty}{\longrightarrow} \varphi_2 \qquad \text{(in point-norm)}$$

and

$$\max_{g\in K} \|\beta_g(v_n) - v_n\| \stackrel{n\to\infty}{\longrightarrow} 0$$

for every compact set $K \subset G$.

Definition

Let $\mathcal{D}$ be a separable, unital C^*-algebra and $\gamma : G \curvearrowright \mathcal{D}$ an action. We say that γ is strongly self-absorbing, if the equivariant first-factor embedding

$$\mathrm{id}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}} : (\mathcal{D}, \gamma) \to (\mathcal{D} \otimes \mathcal{D}, \gamma \otimes \gamma)$$

is approximately G-unitarily equivalent to an isomorphism.

Definition

Let $\mathcal{D}$ be a separable, unital C^*-algebra and $\gamma : G \curvearrowright \mathcal{D}$ an action. We say that γ is strongly self-absorbing, if the equivariant first-factor embedding

$$\mathrm{id}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}} : (\mathcal{D}, \gamma) \to (\mathcal{D} \otimes \mathcal{D}, \gamma \otimes \gamma)$$

is approximately G-unitarily equivalent to an isomorphism.

We recover Toms-Winter's definition of a strongly self-absorbing C^*-algebra by inserting G as the trivial group. Moreover, any $\mathcal{D}$ above must be strongly self-absorbing to begin with.

Definition

Let $\mathcal{D}$ be a separable, unital C^*-algebra and $\gamma : G \curvearrowright \mathcal{D}$ an action. We say that γ is strongly self-absorbing, if the equivariant first-factor embedding

$$\mathrm{id}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}} : (\mathcal{D}, \gamma) \to (\mathcal{D} \otimes \mathcal{D}, \gamma \otimes \gamma)$$

is approximately G-unitarily equivalent to an isomorphism.

We recover Toms-Winter's definition of a strongly self-absorbing C^*-algebra by inserting G as the trivial group. Moreover, any $\mathcal{D}$ above must be strongly self-absorbing to begin with.

Probably the single most important feature of strongly self-absorbing C^*-algebras is that they allow for a McDuff-type theorem that characterizes when some C^*-algebra absorbs them tensorially.

Let us recall:

Definition (Kirchberg, up to small notational difference)

Let A be a C^*-algebra and ω a free filter on $\mathbb{N}$. Recall that

$$A_\omega = \ell^\infty(\mathbb{N}, A) / \left\{ (x_n)_n \mid \lim_{n\to\omega} \|x_n\| = 0 \right\}.$$

Consider

$$A_\omega \cap A' = \{x \in A_\omega \mid [x, A] = 0\}$$

and

$$\mathrm{Ann}(A, A_\omega) = \{x \in A_\omega \mid xA = Ax = 0\}.$$

Notice that $\mathrm{Ann}(A, A_\omega) \subset A_\omega \cap A'$ is an ideal, and one defines

$$F_\omega(A) = A_\omega \cap A' / \mathrm{Ann}(A, A_\omega).$$

Remark

If A is σ-unital, then $F_\omega(A)$ is unital. Overall, the assignment $A \mapsto F_\omega(A)$ is more well-behaved than $A \mapsto A_\omega \cap A'$ or $A \mapsto \mathcal{M}(A)_\omega \cap A'$.

Remark

If A is σ-unital, then $F_\omega(A)$ is unital. Overall, the assignment $A \mapsto F_\omega(A)$ is more well-behaved than $A \mapsto A_\omega \cap A'$ or $A \mapsto \mathcal{M}(A)_\omega \cap A'$.

Remark

If $\alpha : G \curvearrowright A$ is an action, then componentwise application of α on representing sequences yields actions $\alpha_\omega : G \curvearrowright A_\omega$ and $\tilde{\alpha}_\omega : G \curvearrowright F_\omega(A)$.

The following equivariant McDuff-type theorem holds for strongly self-absorbing actions:

Theorem (generalizing Rørdam, Toms-Winter, Kirchberg)

Let $\alpha : G \curvearrowright A$ be an action on a separable C^*-algebra. Let $\gamma : G \curvearrowright \mathcal{D}$ be a strongly self-absorbing action. The following are equivalent:

(1) *α is cocycle conjugate to $\alpha \otimes \gamma$. ($\alpha \simeq_{\mathrm{cc}} \alpha \otimes \gamma$)*

(2) *There exists an equivariant and unital $*$-homomorphism from $(\mathcal{D}, \gamma)$ to $(F_\omega(A), \tilde{\alpha}_\omega)$.*

(3) *There exists an equivariant $*$-homomorphism*

$$\psi : (A \otimes \mathcal{D}, \alpha \otimes \gamma) \to (A_\omega, \alpha_\omega)$$

such that $\psi(a \otimes \mathbf{1}) = a$ for all $a \in A$.

The following equivariant McDuff-type theorem holds for strongly self-absorbing actions:

Theorem (generalizing Rørdam, Toms-Winter, Kirchberg)

Let $\alpha : G \curvearrowright A$ be an action on a separable C^*-algebra. Let $\gamma : G \curvearrowright \mathcal{D}$ be a strongly self-absorbing action. The following are equivalent:

(1) *α is cocycle conjugate to $\alpha \otimes \gamma$. ($\alpha \simeq_{\mathrm{cc}} \alpha \otimes \gamma$)*

(2) *There exists an equivariant and unital $*$-homomorphism from $(\mathcal{D}, \gamma)$ to $(F_\omega(A), \tilde{\alpha}_\omega)$.*

(3) *There exists an equivariant $*$-homomorphism*

$$\psi : (A \otimes \mathcal{D}, \alpha \otimes \gamma) \to (A_\omega, \alpha_\omega)$$

such that $\psi(a \otimes \mathbf{1}) = a$ for all $a \in A$.

The above result also holds for cocycle actions $(\alpha, u) : G \curvearrowright A$. Moreover, cocycle conjugacy **cannot** be strengthened to conjugacy above.

Given a strongly self-absorbing C^*-algebra $\mathcal{D}$, it has been shown by Toms-Winter that $\mathcal{D}$-stability is a property that is closed under various natural C^*-algebraic constructions.

Given a strongly self-absorbing C^*-algebra $\mathcal{D}$, it has been shown by Toms-Winter that $\mathcal{D}$-stability is a property that is closed under various natural C^*-algebraic constructions.

This turns out to generalize to the equivariant situation:

Theorem (generalizing Toms-Winter)

Let $\alpha : G \curvearrowright A$ be an action on a separable C^*-algebra and $\gamma : G \curvearrowright \mathcal{D}$ a strongly self-absorbing action. Assume $\alpha \simeq_{\mathrm{cc}} \alpha \otimes \gamma$. Then:

(1) *If $E \subset A$ is hereditary and α-invariant, then $\alpha|_E \simeq_{\mathrm{cc}} \alpha|_E \otimes \gamma$;*

(2) *If $J \subset A$ is an α-invariant ideal, then $\alpha^{\mathrm{mod} J} \simeq_{\mathrm{cc}} \alpha^{\mathrm{mod} J} \otimes \gamma$;*

(3) *If $\beta : G \curvearrowright B$ and $\delta_i : G \curvearrowright \mathcal{K}$ for $i = 1, 2$ are actions with $\beta \otimes \delta_1 \simeq_{\mathrm{cc}} \alpha \otimes \delta_2$, then $\beta \simeq_{\mathrm{cc}} \beta \otimes \gamma$.*

Theorem (generalizing Toms-Winter)

Let $\gamma : G \curvearrowright \mathcal{D}$ *be a strongly self-absorbing action. If a separable* C^**-dynamical system* (A, α) *arises as an equivariant inductive limit of* C^**-dynamical systems* $(A^{(n)}, \alpha^{(n)})$ *with* $\alpha^{(n)} \simeq_{\mathrm{cc}} \alpha^{(n)} \otimes \gamma$*, then* $\alpha \simeq_{\mathrm{cc}} \alpha \otimes \gamma$.

Theorem (generalizing Toms-Winter)

Let $\gamma : G \curvearrowright \mathcal{D}$ *be a strongly self-absorbing action. If a separable* C^**-dynamical system* (A, α) *arises as an equivariant inductive limit of* C^**-dynamical systems* $(A^{(n)}, \alpha^{(n)})$ *with* $\alpha^{(n)} \simeq_{\mathrm{cc}} \alpha^{(n)} \otimes \gamma$*, then* $\alpha \simeq_{\mathrm{cc}} \alpha \otimes \gamma$*.*

Similar as in Toms-Winter's work, the permanence properties so far are not very hard to prove by using the McDuff-type characterization of γ-absorption. Permanence under extensions, however, is much more challenging.

Theorem (generalizing Toms-Winter)

Let $\gamma : G \curvearrowright \mathcal{D}$ be a strongly self-absorbing action. If a separable C^*-dynamical system (A, α) arises as an equivariant inductive limit of C^*-dynamical systems $(A^{(n)}, \alpha^{(n)})$ with $\alpha^{(n)} \simeq_{\mathrm{cc}} \alpha^{(n)} \otimes \gamma$, then $\alpha \simeq_{\mathrm{cc}} \alpha \otimes \gamma$.

Similar as in Toms-Winter's work, the permanence properties so far are not very hard to prove by using the McDuff-type characterization of γ-absorption. Permanence under extensions, however, is much more challenging.

In the classical setting, a key ingredient in the proof is the fact that unitary commutators are always $\mathbf{1}$-homotopic in a strongly self-absorbing C^*-algebra. We shall discuss an equivariant replacement of this property.

Notation

Let $\alpha : G \curvearrowright A$ an action. Given $\varepsilon > 0$ and a compact set $K \subset G$, we consider the (K, ε)-approximately fixed elements

$$A^{\alpha}_{\varepsilon,K} = \left\{ x \in A \mid \max_{g \in K} \|\alpha_g(x) - x\| \leq \varepsilon \right\}.$$

If A is unital, define

$$\mathcal{U}(A^{\alpha}_{\varepsilon,K}) = A^{\alpha}_{\varepsilon,K} \cap \mathcal{U}(A)$$

and

$$\mathcal{U}_0(A^{\alpha}_{\varepsilon,K}) = \left\{ u \mid \exists\ v : [0,1] \xrightarrow{\text{cont}} \mathcal{U}(A^{\alpha}_{\varepsilon,K}) : v(0) = \mathbf{1}, v(1) = u \right\}.$$

Notation

Let $\alpha : G \curvearrowright A$ an action. Given $\varepsilon > 0$ and a compact set $K \subset G$, we consider the (K, ε)-approximately fixed elements

$$A^{\alpha}_{\varepsilon,K} = \left\{ x \in A \mid \max_{g \in K} \|\alpha_g(x) - x\| \leq \varepsilon \right\}.$$

If A is unital, define

$$\mathcal{U}(A^{\alpha}_{\varepsilon,K}) = A^{\alpha}_{\varepsilon,K} \cap \mathcal{U}(A)$$

and

$$\mathcal{U}_0(A^{\alpha}_{\varepsilon,K}) = \left\{ u \mid \exists\, v : [0,1] \xrightarrow{\text{cont}} \mathcal{U}(A^{\alpha}_{\varepsilon,K}) : v(0) = \mathbf{1}, v(1) = u \right\}.$$

Definition

We call an action $\alpha : G \curvearrowright A$ on a unital C^*-algebra unitarily regular, if for every $\varepsilon > 0$ and compact set $K \subset G$, there exists $\delta > 0$ such that

$$\text{for every } u, v \in \mathcal{U}(A^{\alpha}_{\delta,K}), \ \text{ we have } \quad uvu^*v^* \in \mathcal{U}_0(A^{\alpha}_{\varepsilon,K}).$$

Example

Any action $\alpha : G \curvearrowright A$ with $\alpha \simeq_{\mathrm{cc}} \alpha \otimes \mathrm{id}_{\mathcal{Z}}$ is unitarily regular.

Example

Any action $\alpha : G \curvearrowright A$ with $\alpha \simeq_{\mathrm{cc}} \alpha \otimes \mathrm{id}_{\mathcal{Z}}$ is unitarily regular.

Theorem (generalizing Dadarlat-Winter)

Let $\gamma : G \curvearrowright \mathcal{D}$ be a unitarily regular, strongly self-absorbing action. Let $\alpha : G \curvearrowright A$ be an action on a unital C^*-algebra with $\alpha \simeq_{\mathrm{cc}} \alpha \otimes \gamma$. Then any two equivariant and unital $*$-homomorphisms $\varphi_1, \varphi_2 : (\mathcal{D}, \gamma) \to (A, \alpha)$ are strongly asymptotically G-unitarily equivalent; this means:

Example

Any action $\alpha : G \curvearrowright A$ with $\alpha \simeq_{\mathrm{cc}} \alpha \otimes \mathrm{id}_{\mathcal{Z}}$ is unitarily regular.

Theorem (generalizing Dadarlat-Winter)

Let $\gamma : G \curvearrowright \mathcal{D}$ be a unitarily regular, strongly self-absorbing action. Let $\alpha : G \curvearrowright A$ be an action on a unital C^*-algebra with $\alpha \simeq_{\mathrm{cc}} \alpha \otimes \gamma$. Then any two equivariant and unital $*$-homomorphisms $\varphi_1, \varphi_2 : (\mathcal{D}, \gamma) \to (A, \alpha)$ are strongly asymptotically G-unitarily equivalent; this means:
For every $\varepsilon_0 > 0$ and compact set $K_0 \subset G$, there is a continuous path $w : [0, \infty) \to \mathcal{U}(A)$ satisfying

$$w_0 = \mathbf{1}; \quad \varphi_2 = \lim_{t\to\infty} \mathrm{Ad}(w_t) \circ \varphi_1 \quad \textit{(point-norm)};$$

$$\max_{g\in K} \|\alpha_g(w_t) - w_t\| \stackrel{t\to\infty}{\longrightarrow} 0 \quad \textit{for every compact set } K \subset G;$$

$$\sup_{t\geq 0} \max_{g\in K_0} \|\alpha_g(w_t) - w_t\| \leq \varepsilon_0.$$

Permanence under extensions can then be characterized as follows:

Theorem (generalizing Toms-Winter, Kirchberg)

Let $\gamma : G \curvearrowright \mathcal{D}$ be a strongly self-absorbing action. The following are equivalent:

(1) *The class of all separable, γ-absorbing G-C^*-dynamical systems is closed under extensions.*

(2) *γ is unitarily regular.*

(3) *γ has strongly asymptotically G-inner half-flip.*

(4) *The action $\gamma \star \gamma : G \curvearrowright \mathcal{D} \star \mathcal{D}$ induced on the join is γ-absorbing.*

Reminder: $\mathcal{D} \star \mathcal{D} = \{ f \in \mathcal{C}([0,1], \mathcal{D} \otimes \mathcal{D}) \mid f(0) \in \mathcal{D} \otimes \mathbf{1}, f(1) \in \mathbf{1} \otimes \mathcal{D} \}$

Permanence under extensions can then be characterized as follows:

Theorem (generalizing Toms-Winter, Kirchberg)

Let $\gamma : G \curvearrowright \mathcal{D}$ be a strongly self-absorbing action. The following are equivalent:

(1) *The class of all separable, γ-absorbing G-C^*-dynamical systems is closed under extensions.*

(2) *γ is unitarily regular.*

(3) *γ has strongly asymptotically G-inner half-flip.*

(4) *The action $\gamma \star \gamma : G \curvearrowright \mathcal{D} \star \mathcal{D}$ induced on the join is γ-absorbing.*

Reminder: $\mathcal{D} \star \mathcal{D} = \{ f \in \mathcal{C}([0,1], \mathcal{D} \otimes \mathcal{D}) \mid f(0) \in \mathcal{D} \otimes \mathbf{1}, f(1) \in \mathbf{1} \otimes \mathcal{D} \}$

Question

Is every strongly self-absorbing action unitarily regular?

Example

The trivial G-action on a strongly self-absorbing C^*-algebra $\mathcal{D}$.

Example

The trivial G-action on a strongly self-absorbing C^*-algebra $\mathcal{D}$.

Although this appears uninteresting at first, the equivariant McDuff-type theorem for these actions is a useful tool to verify that certain crossed product C^*-algebras are $\mathcal{D}$-stable.

Example

The trivial G-action on a strongly self-absorbing C^*-algebra $\mathcal{D}$.

Although this appears uninteresting at first, the equivariant McDuff-type theorem for these actions is a useful tool to verify that certain crossed product C^*-algebras are $\mathcal{D}$-stable.

Example

Let D be a separable, unital C^*-algebra with approximately inner flip. Let $u : G \to \mathcal{U}(D)$ be a continuous unitary representation. Then

$$\bigotimes_{\mathbb{N}} \mathrm{Ad}(u) : G \curvearrowright \bigotimes_{\mathbb{N}} D$$

is strongly self-absorbing.

Example

The trivial G-action on a strongly self-absorbing C^*-algebra $\mathcal{D}$.

Although this appears uninteresting at first, the equivariant McDuff-type theorem for these actions is a useful tool to verify that certain crossed product C^*-algebras are $\mathcal{D}$-stable.

Example

Let D be a separable, unital C^*-algebra with approximately inner flip. Let $u : G \to \mathcal{U}(D)$ be a continuous unitary representation. Then

$$\bigotimes_{\mathbb{N}} \mathrm{Ad}(u) : G \curvearrowright \bigotimes_{\mathbb{N}} D$$

is strongly self-absorbing.

This seemingly harmless construction implies the existence of faithful, strongly self-absorbing actions of many groups on various strongly self-absorbing C^*-algebras.

In analogy to the classical situation, one might be tempted to conjecture that all strongly self-absorbing actions are equivariantly $\mathcal{Z}$-stable. This would be an equivariant generalization of a result of Winter.

In analogy to the classical situation, one might be tempted to conjecture that all strongly self-absorbing actions are equivariantly $\mathcal{Z}$-stable. This would be an equivariant generalization of a result of Winter.

Example

The action

$$\gamma = \bigotimes_{\mathbb{N}} \mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} : \mathbb{T} \curvearrowright \bigotimes_{\mathbb{N}} M_2 = M_{2^\infty}$$

is faithful, strongly self-absorbing, but one does **not** have $\gamma \simeq_{\mathrm{cc}} \gamma \otimes \mathrm{id}_{\mathcal{Z}}$.

In analogy to the classical situation, one might be tempted to conjecture that all strongly self-absorbing actions are equivariantly $\mathcal{Z}$-stable. This would be an equivariant generalization of a result of Winter.

Example

The action

$$\gamma = \bigotimes_{\mathbb{N}} \mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} : \mathbb{T} \curvearrowright \bigotimes_{\mathbb{N}} M_2 = M_{2^\infty}$$

is faithful, strongly self-absorbing, but one does **not** have $\gamma \simeq_{\mathrm{cc}} \gamma \otimes \mathrm{id}_{\mathcal{Z}}$. However, γ is unitarily regular.

In analogy to the classical situation, one might be tempted to conjecture that all strongly self-absorbing actions are equivariantly $\mathcal{Z}$-stable. This would be an equivariant generalization of a result of Winter.

Example

The action

$$\gamma = \bigotimes_{\mathbb{N}} \mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} : \mathbb{T} \curvearrowright \bigotimes_{\mathbb{N}} M_2 = M_{2^\infty}$$

is faithful, strongly self-absorbing, but one does **not** have $\gamma \simeq_{\mathrm{cc}} \gamma \otimes \mathrm{id}_{\mathcal{Z}}$. However, γ is unitarily regular.

Next, we shall consider interesting model actions on Kirchberg algebras.

Example

Let G be discrete and exact. By Kirchberg's $\mathcal{O}_2$-embedding theorem, we find a faithful unitary representation $v : G \to \mathcal{U}(\mathcal{O}_2)$. (via $\mathrm{C}^*_r(G) \subset \mathcal{O}_2$) Choose some embedding $\iota : \mathcal{O}_2 \to \mathcal{O}_\infty$, and obtain $u : G \to \mathcal{U}(\mathcal{O}_\infty)$ via

$$u_g = \iota(v_g) + \mathbf{1} - \iota(\mathbf{1}).$$

Example

Consider

$$\delta = \bigotimes_{\mathbb{N}} \mathrm{Ad}(v) : G \curvearrowright \mathcal{O}_2 \quad \text{and} \quad \gamma = \bigotimes_{\mathbb{N}} \mathrm{Ad}(u) : G \curvearrowright \mathcal{O}_\infty.$$

Then both actions are pointwise outer and strongly self-absorbing.

Example

Consider

$$\delta = \bigotimes_{\mathbb{N}} \mathrm{Ad}(v) : G \curvearrowright \mathcal{O}_2 \quad \text{and} \quad \gamma = \bigotimes_{\mathbb{N}} \mathrm{Ad}(u) : G \curvearrowright \mathcal{O}_\infty.$$

Then both actions are pointwise outer and strongly self-absorbing.

Theorem (Izumi, Goldstein-Izumi)

Let G be finite and $\alpha : G \curvearrowright A$ an action on a unital Kirchberg algebra.

(1) $\alpha \otimes \delta$ *is conjugate to* δ.

(2) *if* α *is pointwise outer, then* $\alpha \otimes \mathrm{id}_{\mathcal{O}_2}$ *is conjugate to* δ.

(3) *if* α *is pointwise outer, then* $\alpha \otimes \gamma$ *is conjugate to* α.

Example

Consider

$$\delta = \bigotimes_{\mathbb{N}} \mathrm{Ad}(v) : G \curvearrowright \mathcal{O}_2 \quad \text{and} \quad \gamma = \bigotimes_{\mathbb{N}} \mathrm{Ad}(u) : G \curvearrowright \mathcal{O}_\infty.$$

Then both actions are pointwise outer and strongly self-absorbing.

Theorem (Izumi, Goldstein-Izumi)

Let G be finite and $\alpha : G \curvearrowright A$ an action on a unital Kirchberg algebra.

(1) *$\alpha \otimes \delta$ is conjugate to δ.*

(2) *if α is pointwise outer, then $\alpha \otimes \mathrm{id}_{\mathcal{O}_2}$ is conjugate to δ.*

(3) *if α is pointwise outer, then $\alpha \otimes \gamma$ is conjugate to α.*

Remark

In ongoing work of Phillips on finite group actions on unital Kirchberg algebras, these actions are relevant.

Example

Consider

$$\delta = \bigotimes_{\mathbb{N}} \mathrm{Ad}(v) : G \curvearrowright \mathcal{O}_2 \quad \text{and} \quad \gamma = \bigotimes_{\mathbb{N}} \mathrm{Ad}(u) : G \curvearrowright \mathcal{O}_\infty.$$

Theorem (S)

Let G be discrete, amenable and $\alpha : G \curvearrowright A$ an action on a unital Kirchberg algebra. Then:

(1) $\alpha \otimes \delta \simeq_{\mathrm{cc}} \delta$.

(2) *if α is pointwise outer, then $\alpha \otimes \mathrm{id}_{\mathcal{O}_2} \simeq_{\mathrm{cc}} \delta$.*

(3) *if α is pointwise outer, then $\alpha \otimes \gamma \simeq_{\mathrm{cc}} \alpha$. ($G$ r.f. $\Rightarrow \dim_{\mathrm{Rok}}(\alpha) \leq 1$.)*

(4) $\alpha \otimes \mathrm{id}_{\mathcal{O}_\infty} \simeq_{\mathrm{cc}} \alpha$.

Example

Consider

$$\delta = \bigotimes_{\mathbb{N}} \mathrm{Ad}(v) : G \curvearrowright \mathcal{O}_2 \quad \text{and} \quad \gamma = \bigotimes_{\mathbb{N}} \mathrm{Ad}(u) : G \curvearrowright \mathcal{O}_\infty.$$

Theorem (S)

Let G be discrete, amenable and $\alpha : G \curvearrowright A$ an action on a unital Kirchberg algebra. Then:

(1) $\alpha \otimes \delta \simeq_{\mathrm{cc}} \delta$.

(2) *if α is pointwise outer, then* $\alpha \otimes \mathrm{id}_{\mathcal{O}_2} \simeq_{\mathrm{cc}} \delta$.

(3) *if α is pointwise outer, then* $\alpha \otimes \gamma \simeq_{\mathrm{cc}} \alpha$. *($G$ r.f. $\Rightarrow \dim_{\mathrm{Rok}}(\alpha) \leq 1$.)*

(4) $\alpha \otimes \mathrm{id}_{\mathcal{O}_\infty} \simeq_{\mathrm{cc}} \alpha$.

Question

Can γ and δ be used as cornerstones in some classification theory of outer amenable group actions on Kirchberg algebras?

Thank you for your attention!