Equivariant Kirchberg-Phillips-type absorption for amenable group actions

Workshop C^*-Algebren, Oberwolfach

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As we have seen in earlier talks, an important C^*-algebraic regularity property is given by the tensorial absorption of some strongly self-absorbing C^*-algebra $\mathcal{D}$. This ties into the Toms-Winter conjecture. The most general case concerns $\mathcal{D} = \mathcal{Z}$.

As we have seen in earlier talks, an important C^*-algebraic regularity property is given by the tensorial absorption of some strongly self-absorbing C^*-algebra $\mathcal{D}$. This ties into the Toms-Winter conjecture. The most general case concerns $\mathcal{D} = \mathcal{Z}$.

The earliest and perhaps most prominent case is Kirchberg-Phillips' classification of purely infinite C^*-algebras, where the Cuntz algebra $\mathcal{O}_\infty$ played this role. Together with $\mathcal{O}_2$, which plays a reverse role to $\mathcal{O}_\infty$, these two objects are the cornerstones of this classification theory.

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Theorem (Connes, Jones, Ocneanu, Sutherland-Takesaki, Kawahigashi-Sutherland-Takesaki, Katayama-Sutherland-Takesaki)

Let M be an injective factor and G a discrete amenable group. Then two pointwise outer G-actions on M are cocycle conjugugate by an approximately inner automorphism if and only if they agree on the Connes-Takesaki module.

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More recently, Masuda has found a unified approach for McDuff-factors based on Evans-Kishimoto intertwining. Moreover, there now exist many convincing results of this spirit beyond the discrete group case.

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Motivated by the importance of strongly self-absorbing C^*-algebras in the Elliott program, we ask:

Question

- Is there a dynamical analogue of a strongly self-absorbing C^*-algebra?
- Can we classify C^*-dynamical systems that absorb such objects?

From now, let G denote a second-countable, locally compact group.

Definition

Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ denote actions on separable, unital C*-algebras. Let $\varphi_1, \varphi_2 : (A, \alpha) \to (B, \beta)$ be two equivariant and unital $*$-homomorphisms. We say that φ_1 and φ_2 are approximately G-unitarily equivalent, denoted $\varphi_1 \approx_{\mathrm{u},G} \varphi_2$, if there is a sequence of unitaries $v_n \in B$ with

$$\mathrm{Ad}(v_n) \circ \varphi_1 \stackrel{n\to\infty}{\longrightarrow} \varphi_2 \qquad \text{(in point-norm)}$$

and

$$\max_{g \in K} \|\beta_g(v_n) - v_n\| \stackrel{n\to\infty}{\longrightarrow} 0$$

for every compact set $K \subset G$.

Definition

Let $\mathcal{D}$ be a separable, unital C^*-algebra and $\gamma : G \curvearrowright \mathcal{D}$ an action. We say that γ is strongly self-absorbing, if the equivariant first-factor embedding

$$\mathrm{id}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}} : (\mathcal{D}, \gamma) \to (\mathcal{D} \otimes \mathcal{D}, \gamma \otimes \gamma)$$

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We say that an action $\alpha : G \curvearrowright A$ on a separable C^*-algebra is γ-absorbing, if α is (strongly) cocycle conjugate to $\alpha \otimes \gamma$. (Examples show that demanding conjugacy is unreasonable for non-compact G.)

The following McDuff-type result has been folklore for some time:

Theorem (generalizing Rørdam)

Let G be a countable, discrete group. Let $\alpha : G \curvearrowright A$ be an action on a separable, unital C^*-algebra. Let $\gamma : G \curvearrowright \mathcal{D}$ be a strongly self-absorbing action. Then α is γ-absorbing iff there exists an equivariant and unital $*$-homomorphism from $(\mathcal{D}, \gamma)$ to $(A_\infty \cap A', \alpha_\infty)$.

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Theorem (S, generalizing above, Toms-Winter, Kirchberg)

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To keep this talk more simple, the first theorem will be sufficient for all cases we consider in this talk .

Definition

We call an action $\gamma : G \curvearrowright \mathcal{D}$ semi-strongly self-absorbing, if

$$\mathbf{1}_{\mathcal{D}} \otimes \mathrm{id}_{\mathcal{D}} \approx_{\mathrm{u},G} \mathrm{id}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}}$$

and there exists a unital $*$-homomorphism from $(\mathcal{D}, \gamma)$ to $(\mathcal{D}_\infty \cap \mathcal{D}', \gamma_\infty)$.

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Remark

Unless G is compact, this property is genuinely weaker than strong self-absorption. In general, one only has $\gamma \simeq_{\mathrm{cc}} \gamma \otimes \gamma$ here, with conjugacy iff γ is in fact strongly self-absorbing. We shall also consider this notion because it is sometimes better behaved and easier to verify.

We shall now look at amenable group actions on Kirchberg algebras.

Theorem (Izumi-Matui, unpublished)

Let G be a poly-$\mathbb{Z}$ group and $\mathcal{D}$ a strongly self-absorbing UCT Kirchberg algebra. Then all outer G-actions on $\mathcal{D}$ mutually cocycle conjugate [and in fact semi-strongly self-absorbing]. Moreover, given an outer action $\gamma : G \curvearrowright \mathcal{D}$, every outer action $\alpha : G \curvearrowright A$ on a unital, $\mathcal{D}$-stable Kirchberg algebra is γ-absorbing.

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Corollary

Let G be a poly-$\mathbb{Z}$-group. Then every outer G-action on a unital Kirchberg algebra absorbs any outer G-action on $\mathcal{O}_\infty$, and gets tensorially absorbed by any outer G-action on $\mathcal{O}_2$.

These results are very strong, and in fact only possible because poly-$\mathbb{Z}$ groups are somewhat special. For groups with torsion, this theorem cannot be expected because of K-theoretic obstructions, and in fact already fails for $\mathbb{Z}_2$. However, there are no K-theoretic obstructions for a general result of this kind only involving certain actions on $\mathcal{O}_\infty$ and $\mathcal{O}_2$.

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Theorem (Izumi)

Let G be a finite group. Then up to conjugacy, there exists a unique G-action on $\mathcal{O}_2$ with the Rokhlin property.

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Proposition (not hard to check)

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Corollary

Let G be a finite group. Then every outer G-action on a unital Kirchberg algebra is absorbed by the unique Rokhlin G-action on $\mathcal{O}_2$.

Remark

Let G be a countable, discrete group. Consider $\gamma^{\mathfrak{q}} : G \curvearrowright \mathcal{O}_\infty$,

$$\mathcal{O}_\infty := \mathrm{C}^*(s_{i,g} \mid i \in \mathbb{N},\ g \in G,\ \text{usual relations}),$$

given by $\gamma^{\mathfrak{q}}_g(s_{i,h}) = s_{i,gh}$. This is a typical quasi-free action.

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Theorem (Goldstein-Izumi, Phillips)

Let G be a finite group. Then $\gamma^{\mathfrak{q}}$ is strongly self-absorbing, and is absorbed by every outer G-action on a unital Kirchberg algebra.

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Problem: The proofs of these results use techniques only available to the respective classes of acting groups. How do we circumvent them?

Example

Let D be a separable, unital C^*-algebra with approximately inner flip. Let $u : G \to \mathcal{U}(D)$ be a continuous unitary representation of a locally compact group. Then

$$\bigotimes_{\mathbb{N}} \mathrm{Ad}(u) : G \curvearrowright \bigotimes_{\mathbb{N}} D$$

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Example

Let G be discrete and exact. By Kirchberg's $\mathcal{O}_2$-embedding theorem, we find a faithful unitary representation $v : G \to \mathcal{U}(\mathcal{O}_2)$. (via $\mathrm{C}^*_r(G) \subset \mathcal{O}_2$) Choose some embedding $\iota : \mathcal{O}_2 \to \mathcal{O}_\infty$, and obtain $u : G \to \mathcal{U}(\mathcal{O}_\infty)$ via $u_g = \iota(v_g) + \mathbf{1} - \iota(\mathbf{1})$. Consider

$$\delta = \bigotimes_{\mathbb{N}} \mathrm{Ad}(v) : G \curvearrowright \bigotimes_{\mathbb{N}} \mathcal{O}_2 \cong \mathcal{O}_2 \ , \ \gamma = \bigotimes_{\mathbb{N}} \mathrm{Ad}(u) : G \curvearrowright \bigotimes_{\mathbb{N}} \mathcal{O}_\infty \cong \mathcal{O}_\infty.$$

Theorem (Izumi, Goldstein-Izumi)

Let G be a finite group. Then:

(1) *For any outer action $\alpha : G \curvearrowright A$ on a unital Kirchberg algebra, $\alpha \otimes \mathrm{id}_{\mathcal{O}_2}$ is a Rokhlin action.*

(2) *δ is a Rokhlin action.*

(3) *γ is conjugate to $\gamma^{\mathfrak{q}}$.*

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Idea: Use these actions as candidates for an absorption theorem for all amenable groups.

Reminder

$$\delta = \bigotimes_{\mathbb{N}} \mathrm{Ad}(v) : G \curvearrowright \mathcal{O}_2 \quad , \quad \gamma = \bigotimes_{\mathbb{N}} \mathrm{Ad}(u) : G \curvearrowright \mathcal{O}_\infty .$$

Let us first consider the absorbing object δ.

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Let us first consider the absorbing object δ.

Theorem (S)

Let G be a discrete, amenable group. Then up to (strong) cocycle conjugacy, δ is the unique outer, equivariantly $\mathcal{O}_2$-absorbing G-action on $\mathcal{O}_2$. In particular, we have $\alpha \otimes \delta \simeq_{\mathrm{cc}} \delta$ for any action $\alpha : G \curvearrowright A$ on a unital Kirchberg algebra.

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To my knowledge, this marks the first C^*-algebraic result of this kind for actions that is applicable to all amenable groups.

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Let G be a discrete, amenable, residually finite group. Then every outer G-action on a unital Kirchberg algebra has Rokhlin dimension at most one.

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Question

Can γ be characterized abstractly?

Theorem (S)

Let G be a discrete, amenable group. Let $\beta : G \curvearrowright \mathcal{O}_\infty$ be an outer action. Then β is strongly cocycle conjugate to γ iff

- *the inclusion $\mathrm{C}^*(G) \subset \mathcal{O}_\infty \rtimes_\beta G$ is a KK-equivalence; (or: KL)*
- *β is approximately representable. (or: has $\approx G$-inner half-flip)*

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(A variant of this holds for every ssa Kirchberg algebra $\mathcal{D}$.)

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The first condition must generally be assumed for groups with torsion. It remains unclear whether the second one is always redundant.

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It would be interesting and natural to find out whether these conditions hold for Bernoulli shifts or quasi-free actions.

Theorem (torsion-free case; using Baum-Connes)

Let G be a discrete, amenable, torsion-free group and $\mathcal{D}$ a ssa Kirchberg algebra. Then up to (strong) cocycle conjugacy, $\gamma \otimes \mathrm{id}_\mathcal{D}$ is the unique outer, approximately representable G-action on $\mathcal{D}$.

Thank you for your attention!