Rokhlin dimension and topological dynamics

Asymptotic decomposition methods in geometry, dynamics and operator algebras

Gábor Szabó

University of Aberdeen

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It all starts with the classical Rokhlin Lemma in measurable dynamics:

Theorem (Rokhlin, 1930s)

Let (X, μ) *be a standard Borel probability space. Let* $T : X \to X$ *be a measure-preserving, invertible, aperiodic transformation. Then for every* $n \in \mathbb{N}$ *and* $\varepsilon > 0$*, there exists a Borel set* $E \subset X$ *such that*

$$E \cap T^j(E) = \emptyset \quad \textit{for all } j = 1, \ldots, n-1$$

and

$$\mu\Big(\bigcup_{j=0}^{n-1} T^j(E)\Big) \geq 1 - \varepsilon.$$

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It quickly found its way into C^*-algebra theory. The following is the most naive definition, although not the standard one today:

Definition

Let A be a separable, unital C^*-algebra and $\psi : A \to A$ an automorphism. We say that ψ has the strict Rokhlin property, if for every $n \in \mathbb{N}$, $\varepsilon > 0$ and $\mathcal{F} \Subset A$, there exist projections $\{p_j\}_{j=0,\dots,n-1}$ with:

- $\mathbf{1} = \sum_{j=0}^{n-1} p_j$;
- $\|\psi(p_j) - p_{j+1}\| \leq \varepsilon$ for all $j = 0, \dots, n-1$; (denoting $p_n = p_0$)
- $\|ap_j - p_j a\| \leq \varepsilon$ for all j and $a \in \mathcal{F}$.

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After its inception, this property was used for hard classification results of automorphisms on simple C^*-algebras.

Proposition

$\psi \in \mathrm{Aut}(A)$ has the strict Rokhlin property if and only if for every $n \in \mathbb{N}$ there exists an equivariant and unital $*$-homomorphism

$$(\mathcal{C}(\mathbb{Z}/n\mathbb{Z}), \mathbb{Z}\text{-shift}) \to (A_\infty \cap A', \psi_\infty).$$

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Example

A homeomorphism $T : X \to X$ on a compact space gives rise to an automorphism on $\mathcal{C}(X)$ with the strict Rokhlin property if and only if for every $n \in \mathbb{N}$, there exists an equivariant factor map

$$\beta(\mathbb{N} \times X) \setminus (\mathbb{N} \times X) \to \mathbb{Z}/n\mathbb{Z}.$$

Definition (Winter-Zacharias)

A completely positive map $\varphi : A \to B$ between C^*-algebras is said to be of order zero, if whenever $a, b \in A$ are positive elements with $ab = 0$, then $\varphi(a)\varphi(b) = 0$.

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Definition (Winter-Zacharias)

A C^*-algebra A is said to have nuclear dimension at most $n \geq 0$, written $\dim_{\mathrm{nuc}}(A) \leq n$, if for every $\mathcal{F} \Subset A$ and $\varepsilon > 0$ there exist

- a finite-dimensional C^*-algebra F;
- a completely positive and contractive map $\psi : A \to F$;
- completely positive, contractive order zero maps $\varphi^{(0)}, \ldots, \varphi^{(n)} : F \to A$

such that

$$\|a - \sum_{l=0}^{n} (\varphi^{(l)} \circ \psi)(a)\| \leq \varepsilon \quad \text{for all } a \in \mathcal{F}.$$

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For a locally compact, metrizable space X, one has

$$\dim_{\mathrm{nuc}}(\mathcal{C}_0(X)) = \dim(X).$$

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Theorem (Kirchberg-Phillips, ..., (Elliott-)Gong-Lin-Niu, Tikuisis-White-Winter)

Let A and B be two separable, simple, unital C^*-algebras with finite nuclear dimension satisfying the UCT. Then A and B are isomorphic if and only if they agree on K-theory and traces.

Definition

Let $\alpha : G \curvearrowright A$ be an action of a discrete group G on a C^*-algebra A by automorphisms. The crossed product of (A, α, G) is the universal C^*-algebra

$$A \rtimes_\alpha G = \mathrm{C}^*\Big(A \cup \{\lambda_g\}_{g\in G} \mid [g \mapsto \lambda_g] \text{ is a unitary rep., } \lambda_g a \lambda_g^* = \alpha_g(a)\Big).$$

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For an action $G \curvearrowright X$ on a compact space, one can associate $\alpha : G \curvearrowright \mathcal{C}(X)$ via $\alpha_g(f)(x) = f(g^{-1}.x)$. The crossed product $\mathcal{C}(X) \rtimes_\alpha G$ is a C^*-algebra that encodes the underlying topological dynamical system in its noncommutative structure.

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Theorem (Hirshberg-Winter-Zacharias)

If an automorphism $\psi \in \mathrm{Aut}(A)$ *has the strict Rokhlin property and* $\dim_{\mathrm{nuc}}(A) < \infty$, *then* $\dim_{\mathrm{nuc}}(A \rtimes_\psi \mathbb{Z}) < \infty$.

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Their methods to prove this were so flexible that this gave rise to a much more flexible notion.

Definition (Hirshberg-Winter-Zacharias)

For an automorphism $\psi \in \mathrm{Aut}(A)$, the Rokhlin dimension $d = \dim_{\mathrm{Rok}}(\psi)$ is the smallest natural number such that for each n, there exist equivariant order zero maps

$$\varphi^{(0)}, \ldots, \varphi^{(d)} : (\mathcal{C}(\mathbb{Z}/n\mathbb{Z}), \mathbb{Z}\text{-shift}) \to (A_\infty \cap A', \psi_\infty)$$

such that $\varphi^{(0)}(\mathbf{1}) + \cdots + \varphi^{(d)}(\mathbf{1}) = \mathbf{1}$.

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Advantage: This avoids many severe obstructions that the strict Rokhlin property has. (K-theory, absence of projections,...)

Theorem (Hirshberg-Winter-Zacharias)

For any automorphism $\psi \in \mathrm{Aut}(A)$*, one has*

$$\dim_{\mathrm{nuc}}^{+1}(A \rtimes_{\psi} \mathbb{Z}) \leq 2 \dim_{\mathrm{Rok}}^{+1}(\psi) \dim_{\mathrm{nuc}}^{+1}(A).$$

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Example

A homeomorphism $T : X \to X$ on a compact space gives rise to an automorphism on $\mathcal{C}(X)$ with Rokhlin dimension at most d if and only if for every $n \in \mathbb{N}$, there exists an equivariant continuous map

$$\beta(\mathbb{N} \times X) \setminus (\mathbb{N} \times X) \to \underbrace{(\mathbb{Z}/n\mathbb{Z}) \star \cdots \star (\mathbb{Z}/n\mathbb{Z})}_{(d+1)\text{-times}}.$$

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Problem: In practice, this is absolutely useless. Nevertheless, there are at least sufficient criteria that are useful. (Similar to tower dimension)

Theorem (Hirshberg-Winter-Zacharias, S)

If $T : X \to X$ is an aperiodic homeomorphism on a compact metric space X with $\dim(X) < \infty$, then the induced automorphism on $\mathcal{C}(X)$ has finite Rokhlin dimension. In particular, $\dim_{\mathrm{nuc}}(\mathcal{C}(X) \rtimes_T \mathbb{Z}) < \infty$.

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Definition (S-Wu-Zacharias)

Let G be a discrete group and $H \leq G$ with $[G : H] < \infty$. Let $\alpha : G \curvearrowright A$ be an action on a separable, unital C^*-algebra. We say that α has Rokhlin dimension at most $d \geq 0$ relative to H, written $\dim_{\mathrm{Rok}}(\alpha, H) \leq d$, if there exist equivariant order zero maps

$$\varphi^{(0)}, \ldots, \varphi^{(d)} : (\mathcal{C}(G/H), G\text{-shift}) \to (A_\infty \cap A', \alpha_\infty)$$

such that $\varphi^{(0)}(\mathbf{1}) + \cdots + \varphi^{(d)}(\mathbf{1}) = \mathbf{1}$.

Definition (continued)

Now assume G is residually finite. Let $\sigma = (G_n)_n$ be a decreasing sequence of normal, finite-index subgroups with trivial intersection. The Rokhlin dimension of α along σ is

$$\dim_{\mathrm{Rok}}(\alpha, \sigma) = \sup_{n \in \mathbb{N}} \dim_{\mathrm{Rok}}(\alpha, G_n).$$

Lastly, we define the full Rokhlin dimension of α as

$$\dim_{\mathrm{Rok}}(\alpha) = \sup \left\{ \dim_{\mathrm{Rok}}(\alpha, H) \mid H \leq G,\ [G : H] < \infty \right\}.$$

Definition

Let G be a residually finite group. For a residually finite approximation $\sigma = (G_n)_n$ as before, consider the box space of G along σ as the coarse disjoint union

$$\Box_\sigma G = \bigsqcup_{n \in \mathbb{N}} G/G_n.$$

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Theorem (S-Wu-Zacharias)

Let G be a residually finite group and choose σ as before. For an action $\alpha : G \curvearrowright A$, one has

$$\dim_{\mathrm{nuc}}^{+1}(A \rtimes_\alpha G) \leq \mathrm{asdim}^{+1}(\Box_\sigma G)\, \dim_{\mathrm{Rok}}^{+1}(\alpha,\sigma)\, \dim_{\mathrm{nuc}}^{+1}(A).$$

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Finite asymptotic dimension implies property A.

Example

- One can find σ for G with $\mathrm{asdim}(\Box_\sigma G) < \infty$ when G is finitely generated and virtually nilpotent. (S-Wu-Zacharias)
- Same when G is virtually poly-cyclic. (Finn-Sell-Wu)
- In such cases, $\mathrm{asdim}(\Box_\sigma G) = \mathrm{asdim}(G)$. (Yamauchi, apparently)

Notation

For a free action $\alpha : G \curvearrowright X$ on a compact space, we say that α has amenability dimension $d \geq 0$ if d is the smallest number such that α is $(d+1)$-amenable in the sense of Guentner-Willett-Yu.

Note: $\mathrm{dad}(\alpha) \leq \dim_{\mathrm{am}}(\alpha)$.

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Theorem (Guentner-Willett-Yu)

For a free action $\alpha : G \curvearrowright X$, one has

$$\dim_{\mathrm{nuc}}^{+1}(\mathcal{C}(X) \rtimes_\alpha G) \leq \dim_{\mathrm{am}}^{+1}(\alpha) \dim^{+1}(X).$$

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Theorem (S-Wu-Zacharias)

For a given pair (G, σ) and a free action $\alpha : G \curvearrowright X$, one has

$$\dim_{\mathrm{Rok}}^{+1}(\alpha) \leq \dim_{\mathrm{am}}^{+1}(\alpha) \leq \mathrm{asdim}^{+1}(\Box_\sigma G) \dim_{\mathrm{Rok}}^{+1}(\alpha, \sigma).$$

Example

For $\sigma = (G_n)_n$, one has $\dim_{\mathrm{am}}\left(G \curvearrowright \varprojlim G/G_n\right) = \operatorname{asdim}(\Box_\sigma G)$.

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Theorem (S, S-Wu-Zacharias, Bartels)

Let G be a finitely generated, virtually nilpotent group. Let $\alpha : G \curvearrowright X$ be a free action. If $\dim(X) < \infty$, then $\dim_{\mathrm{Rok}}(\alpha) < \infty$ and $\dim_{\mathrm{am}}(\alpha) < \infty$.

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Corollary

If in the above situation, α is also minimal, then $\mathcal{C}(X) \rtimes_\alpha G$ is classifiable.

Notation

A flow on a C^*-algebra A is a continuous action $\alpha : \mathbb{R} \curvearrowright A$.
If $\Phi : \mathbb{R} \curvearrowright X$ is a topological flow on a compact space, then one obtains a C^*-flow $\alpha : \mathbb{R} \curvearrowright \mathcal{C}(X)$ via $\alpha_t(f) = f \circ \Phi_{-t}$.

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Definition (Hirshberg-S-Winter-Wu)

For a flow $\alpha : \mathbb{R} \curvearrowright A$ on a separable, unital C^*-algebra, the Rokhlin dimension of α is the smallest natural number $d \geq 0$ such that for every $T > 0$ there exist equivariant order zero maps

$$\varphi^{(0)}, \ldots, \varphi^{(d)} : (\mathcal{C}(\mathbb{R}/T\mathbb{Z}), \mathbb{R}\text{-shift}) \to (A_\infty \cap A', \alpha_\infty)$$

such that $\varphi^{(0)}(\mathbf{1}) + \cdots + \varphi^{(d)}(\mathbf{1}) = \mathbf{1}$.

Example

A topological flow $\Phi : \mathbb{R} \curvearrowright X$ on a compact space gives rise to a C^*-flow on $\mathcal{C}(X)$ with Rokhlin dimension at most d if and only if for every $T > 0$, there exists an equivariant continuous map

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As before, there is a more useful sufficient topological criterium to get finite Rokhlin dimension, called finite tube dimension. Essentially, this is the continuous analog of tower dimension from this morning.

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To put it simply without details, finite tube dimension means that the flow Φ admits *long thin covers* similarly as it appears in the work of Bartels-Lück-Reich.

Theorem (Hirshberg-S-Winter-Wu)

For a flow $\alpha : \mathbb{R} \curvearrowright A$, *one has*

$$\dim_{\mathrm{nuc}}^{+1}(A \rtimes_\alpha \mathbb{R}) \leq 2 \dim_{\mathrm{Rok}}^{+1}(\alpha) \dim_{\mathrm{nuc}}^{+1}(A).$$

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Using some techniques in the work of Bartels-Lück-Reich and later improvements by Kasprowski-Rüping, one can show:

Theorem (Hirshberg-S-Winter-Wu)

For a free topological flow $\Phi : \mathbb{R} \curvearrowright X$ *with* $\dim(X) < \infty$*, one has* $\dim_{\mathrm{Rok}}(\Phi) < \infty$*. In particular,* $\dim_{\mathrm{nuc}}(\mathcal{C}(X) \rtimes_\Phi \mathbb{R}) < \infty$*.*

In some cases, this leads to the classification of the crossed products $\mathcal{C}(X) \rtimes_\Phi \mathbb{R}$.

Thank you for your attention!