Introduction to the classification of group actions on C*-algebras

Gábor Szabó

Abstract

These notes serve as supplementary material for a 3-hour lecture series presented at the 16th Spring Institute for Noncommutative Geometry and Operator Algebras (NCGOA), from the 14th to the 19th of May 2018.

The plan of this lecture series is to give an introduction into some of the core ideas leading to the classification of single automorphisms on C*-algebras up to cocycle conjugacy. The emphasis shall be on the key methods and techniques, which will culminate in a master plan of sorts dictated by the past work of Kishimoto and others. More specifically, the plan is to discuss:

- the Rokhlin property for automorphisms;
- approximate cohomology vanishing as a consequence of the Rokhlin property;
- the Evans–Kishimoto intertwining argument.

From a practical point of view, this introduction is intended to be a gentle one, which will lead us to make special assumptions along the way in order to make some proofs more palatable. Nevertheless, the level of generality shall be high enough to arrive at some interesting statements, for example Kishimoto's theorem that there is a unique Rokhlin automorphism on every infinite-dimensional UHF algebra. If time permits, we may even end up proving a theorem together which goes beyond what can be found in the present literature.

The notes are purposefully written in far greater detail than what will be presented in the lectures; the introduction given here is even exclusive to the written material. Please beware that only little proof-reading has been done for these notes.

Introduction

There are two main classes of objects in the theory of operator algebras, namely C*-algebras and von Neumann algebras. As we know from the Gelfand–Naimark theorem, a commutative C*-algebra can be naturally expressed as $\mathcal{C}_0(X)$ for some locally compact Hausdorff space X. C*-algebras are therefore sometimes intuitively regarded as noncommutative topological spaces, while von Neumann algebras are regarded as noncommutative measure spaces for a similar reason. This analogy is helpful for understanding the difference between the two classes, and to view the theory of group actions on C*-algebras and von Neumann algebras as noncommutative generalizations of topological dynamics and ergodic theory, respectively.

An impressive application of noncommutative dynamical systems is given within the Connes–Haagerup classification of injective factors, which in part involves the classification of cyclic group actions on certain factors; see [2, 3, 4, 5]. In part guided by such applications, group actions on operator algebras have been of recurring and great interest in the field. A far reaching generalization of Connes' classification of cyclic group actions has been accomplished by many hands, and we now know that countable amenable group actions on injective factors are completely classified up to cocycle conjugacy by certain computable invariants; see [15, 34, 36, 18, 16] and in particular [25, 26] for a unified treatment which happens to be in line with this lecture series. Group actions on C*-algebras, on the other hand, offer more complicated and interesting structure, but also pose a greater challenge with respect to their classification.

Definition. Fix a locally compact group G. Let $\alpha : G \curvearrowright A$ be a point-norm continuous action on a C*-algebra.

(1) An α-cocycle is a strictly continuous map $w : G \to \mathcal{U}(\mathcal{M}(A))$ satisfying the cocycle identity $w_{gh} = w_g\alpha_g(w_h)$ for all $g, h \in G$.[1]

(2) Let $\beta : G \curvearrowright B$ be some other action. One says that α and β are cocycle conjugate, if there exists an isomorphism $\varphi : A \to B$ and an α-cocycle w such that

$$\mathrm{Ad}(w_g) \circ \alpha_g = \varphi^{-1} \circ \beta_g \circ \varphi, \quad g \in G.$$

[1]The so-called coboundaries are those cocycles that emerge from a single unitary via the formula $w_g = v\alpha_g(v^*)$. These are in a sense considered trivial.

In general, the classification of G-actions on a C*-algebra up to cocycle conjugacy is a very difficult task, even when restricting to more special subclasses of G-actions. Consulting the literature, a pattern emerges which is common to many successful solutions of this problem; see for example [19, 7, 20, 21, 32, 12, 17, 27, 28, 29, 13, 35, 37]. This culminates in a *master plan* of sorts invented by Kishimoto, which can be sketched as follows:[2]

> Suppose we have two actions $\alpha, \beta : G \curvearrowright A$, for which we want to show that they are cocycle conjugate.
>
> **S1:** Show that α and β satisfy some kind of Rokhlin-type property.[3]
>
> **S2:** Exploiting the first step, achieve the following two things:
>
> > **S2.a:** Show that there are α-cocycles w such that $\mathrm{Ad}(w_g) \circ \alpha_g \approx \beta_g$ holds approximately in point-norm over a large finite set in A, and uniformly over a compact set in G. Do the same in the reverse direction, exchanging the roles of α and β.
> >
> > **S2.b:** Show that α has the approximately central *cohomology vanishing* property: For every α-cocycle w with $[a, w_g] \approx 0$ over some large finite set, find a unitary v with $[v, a] \approx 0$ and $w_g \approx v\alpha_g(v^*)$. Do the same for β.
>
> **S3:** Combining the previous steps, apply the Evans–Kishimoto intertwining technique to achieve the desired outcome.

Without already knowing a lot of the relevant literature, the above *recipe* may not be particularly illuminating at first. The point of this lecture series is to introduce the reader / audience to this approach by seeing part of it in action. In order to make this introduction as gentle as possible without becoming uninteresting, we shall restrict our attention to single automorphisms[4], and add some assumptions along the way to save us from an overwhelming amount of complicated setup.

[2] I would like to emphasize that this is only a rough and naive recipe, which often needs further refinement to obtain interesting new results going beyond the state-of-the-art.

[3] For example, this may be automatic from some natural condition like *outerness*, but is often highly non-trivial to show.

[4] Read: actions of $\mathbb{Z}$

In particular, Step S2.a above becomes redundant for single automorphisms, as cocycles with respect to $\mathbb{Z}$-actions are nothing but single unitaries.[5] From this point of view, Step S2.a asks for single automorphisms to be approximately unitarily equivalent to each other, which is a separate problem we may outsource to classification theory of C*-algebras.

Moreover, we will in this lecture series completely disregard the above Step S1, i.e., how to obtain the Rokhlin property from a priori more natural outerness conditions. While this is a very interesting topic, it could easily fill its own lecture series, and may be touched upon by other lectures during the conference. Instead we will always assume the Rokhlin property and see how it can be used to end up with hard classification results.

To summarize, the two key techniques communicated in this lecture series will be how to achieve Steps S2.b and S3 above for single automorphisms.

1 The Rokhlin property

Definition 1.1. Let A and B be two C*-algebras. Suppose that α is an automorphism on A, and that β is an automorphism on B. One says that α and β are cocycle conjugate, if there exists an isomorphism $\varphi : A \to B$ and a unitary $w \in \mathcal{U}(\mathcal{M}(A))$ such that $\mathrm{Ad}(w) \circ \alpha = \varphi^{-1} \circ \beta \circ \varphi$.[6]

Definition 1.2. Let α be an automorphism on a C*-algebra A. A unitary $u \in \mathcal{U}(\mathbf{1} + A)$ is called a coboundary, if it can be expressed as $u = v\alpha(v^*)$ for some $v \in \mathcal{U}(\mathbf{1} + A)$.

Definition 1.3. Let A be a separable, unital C*-algebra.[7] Let α be an automorphism on A. We say that it has the Rokhlin property, if for every $n \in \mathbb{N}$ there exist approximately central sequences of projections $e_k, f_k \in A$ such that

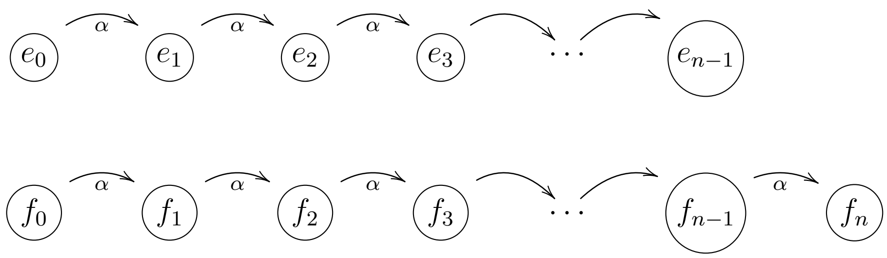
$$\mathbf{1} = \lim_{k\to\infty} \sum_{j=0}^{n-1} \alpha^j(e_k) + \sum_{l=0}^{n} \alpha^l(f_k).$$

[5] Given a single unitary u, one obtains its associated α-cocycle by defining $u_n = u\alpha(u)\cdots\alpha^{n-1}(u)$ for $n \geq 0$ and a similar formula for $n < 0$.

[6] In the case of single automorphisms, one may also call this *outer conjugacy*. For other types of dynamical systems, that usually means something weaker than cocycle conjugacy.

[7] Unitality is for convenience only; the non-unital version involves approximate behavior in the strict topology, or the corrected central sequence algebra $(A_\infty \cap A')/(A_\infty \cap A^\perp)$.

Picture for the relation in terms of $e, f \in A_\infty \cap A'$:
$(e_j := \alpha^j(e), f_l := \alpha^l(f))$



For convenience we will sometimes consider the *strict* Rokhlin property, which means that one may always choose $f_k = 0$ above.[8]

Example 1.4. Let X be a Cantor set. A homeomorphism φ on X is aperiodic if and only if the induced automorphism on $\mathcal{C}(X)$ has the Rokhlin property.

Example 1.5. Every infinite-dimensional UHF algebra admits a Rokhlin automorphism.[9]

Proof. For a fixed $n \in \mathbb{N}$, we consider the direct sum $M_n \oplus M_{n+1}$, and observe that the unitary

$$s_n = \begin{pmatrix} 0 & 1 & \dots & & 0 \\ & 0 & 1 & \dots & \\ \vdots & & \ddots & \ddots & \\ \vdots & & & \ddots & 1 \\ 1 & 0 & \dots & & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & \dots & & & 0 \\ & 0 & 1 & & & \vdots \\ \vdots & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ \vdots & & & & \ddots & 1 \\ 1 & 0 & \dots & & & 0 \end{pmatrix}$$

---

[8] Historically, variants of this have been the first type of Rokhlin property considered by Herman–Ocneanu, but it is genuinely stronger than the modern definition. For example, in contrast to Example 1.5, the strict Rokhlin property in this sense can only occur for the universal UHF algebra.

[9] More generally, the argument presented here can be adapted to show that every unital approximately divisible C*-algebra admits an asymptotically inner automorphism with the Rokhlin property. This is a bit more delicate, so we omit the details.

defines an inner automorphism for which the (standard) minimal projections $e_0, \dots, e_{n-1} \in M_n \oplus 0$ and $f_0, \dots, f_n \in 0 \oplus M_{n+1}$ satisfy the relation in Definition 1.3 on the nose, minus the approximate centrality.

Now let $\mathbb{U}$ be an infinite-dimensional UHF algebra. Then we may identitfy

$$\mathbb{U} \cong \bigotimes_{n,k\in\mathbb{N}} \mathbb{U}_{n,k},$$

where for each $n, k \in \mathbb{N}$, the C*-algebra $\mathbb{U}_{n,k}$ is also an infinite-dimensional UHF algebra. For all $n, k \in \mathbb{N}$, we can find some unital $*$-homomorphism $\iota_{n,k} : M_n \oplus M_{n+1} \to \mathbb{U}_{n,k}$.[10] Under the above identification, we define

$$\alpha = \bigotimes_{n,k\in\mathbb{N}} \mathrm{Ad}(\iota_{n,k}(s_n)) \ \in \ \mathrm{Aut}(\mathbb{U}),$$

which will satisfy the Rokhlin property by construction. □

Theorem 1.6 (Kishimoto [19])**.** *Suppose A is an infinite-dimensional UHF algebra and α an automorphism on A such that the crossed product $A \rtimes_\alpha \mathbb{Z}$ has a unique tracial state. Then α has the Rokhlin property.*

Theorem 1.7 (Nakamura [32])**.** *Suppose A is a Kirchberg algebra and α an automorphism on A which is aperiodic.*[11] *Then α has the Rokhlin property.*

2 Approximate cohomology vanishing

In this section, we go over how to achieve Step S2.b in the approach outlined in the introduction, at least for single automorphisms satisfying the Rokhlin property. In order to make the proofs more palatable, let us make the following strong assumption on our C*-algebras from this point forward:

Notation 2.1. Let A be a C*-algebra. For the rest of this lecture series, we will say that A satisfies property $(\star)$, if there exists some positive constant $L > 0$ such that the following holds:

[10] Every sufficiently big number N can be realized as $N = an + b(n+1)$ for natural numbers $a, b \geq 0$. Thus there is some N such that $M_n \oplus M_{n+1} \subset M_N \subset \mathbb{U}_{n,k}$ unitally, using that the latter must contain matrix algebras of arbitrarily large size.

[11] This means that α^j is outer for $j \neq 0$.

For all $\varepsilon > 0$ and $\mathcal{F} \Subset A$, there exist $\delta > 0$ and $\mathcal{G} \Subset A$ such that for all unitaries $u \in \mathcal{U}(\mathbf{1} + A)$ satisfying

$$\max_{a \in \mathcal{G}} \|[u, a]\| \leq \delta,$$

there exists an L-Lipschitz path $u : [0, 1] \to \mathcal{U}(\mathbf{1} + A)$ satisfying

$$u_0 = \mathbf{1}, \quad u_1 = u, \quad \text{and} \quad \max_{a \in \mathcal{F}} \max_{0 \leq t \leq 1} \|[u_t, a]\| \leq \varepsilon.$$

Remark 2.2. In terms of sequence algebras, property $(\star)$ just means that the unitary group $\mathcal{U}\big(\mathbf{1} + (A_\infty \cap A')\big)$ is connected.

Example 2.3. All AF C*-algebras satisfy property $(\star)$. Moreover, every C*-algebra A with $A \cong A \otimes \mathcal{O}_2$ satisfies property $(\star)$.

Proof. The case $A \cong A \otimes \mathcal{O}_2$ follows from a well-known argument of Haagerup–Rørdam [10, Lemma 5.1]; see [37, Section 5] for a slightly more distilled version of the argument.

The case of AF algebras is well-known. Let us give a sketch of proof for UHF algebras $A = \mathbb{U}$: Suppose $\mathcal{F} \Subset \mathbb{U}$ is given. Then, due to the UHF structure, there is a tensor decomposition $\mathcal{U} \cong M_p \otimes \mathbb{U}_1$ such that $\mathcal{F}$ is approximately contained in $M_p \otimes \mathbf{1}$. So we may as well assume $\mathcal{F} \subseteq M_p \otimes \mathbf{1}$. Let $\mathcal{G}$ be the finite set of matrix units generating this copy of $M_p \otimes \mathbf{1}$. Then it is an easy exercise to see that any unitary (in fact any element), which approximately commutes with the elements in $\mathcal{G}$ sufficiently well, is close to $\mathbf{1}_{M_p} \otimes \mathbb{U}_1$. Thus if $\delta > 0$ is small enough, then every $(\delta, \mathcal{G})$-approximately central unitary $u \in \mathbb{U}$ can be continuously perturbed to a unitary in $\mathbf{1} \otimes \mathbb{U}_1$ with finite spectrum via a short path, where in turn it can be π-Lipschitz connected to the unit in $\mathbf{1} \otimes \mathbb{U}_1$, preserving $(\varepsilon, \mathcal{F})$-approximate centrality. □

The following argument is due to Herman–Ocneanu [11], with subsequent adaptations and refinements due to Kishimoto and many others.

Lemma 2.4. *Let A be a separable* C*-*algebra with property $(\star)$. Let α be an automorphism on A with the Rokhlin property. Then for every $\varepsilon > 0$ and $\mathcal{F} \Subset A$, there exists $\delta > 0$ and $\mathcal{G} \Subset A$ such that whenever $u \in \mathcal{U}(\mathbf{1} + A)$ is a unitary satisfying*

$$\max_{a \in \mathcal{G}} \|[u, a]\| \leq \delta,$$

then there exists a unitary $v \in \mathcal{U}(\mathbf{1} + A)$ satisfying

$$\|u - v\alpha(v^*)\| \leq \varepsilon \quad \text{and} \quad \max_{a \in \mathcal{F}} \|[v, a]\| \leq \varepsilon.$$

Remark. During the lectures, the conclusion of this lemma was referred to as the *one-cocycle property* for α.

Proof. For convenience, let us assume that A is unital and that α has the strict Rokhlin property. In terms of sequence algebras, the claim just means that every unitary $u \in A_\infty \cap A'$ can be expressed as a coboundary $u = v\alpha(v^*)$ for some unitary $v \in A_\infty \cap A'$.

So let $u \in A_\infty \cap A'$ be given. It is enough to show that, given $\varepsilon > 0$, we may find v such that $\|u - v\alpha(v^*)\| \leq \varepsilon$. So let us also fix $\varepsilon > 0$ for the rest of the proof. Let $L > 0$ be the constant supplied to us by property $(\star)$. We choose $n \in \mathbb{N}$ with $\frac{L}{n} \leq \varepsilon$.

The unitary u corresponds to the cocycle given by the formula $u_k := u\alpha(u)\cdots\alpha^{k-1}(u)$ for $k \geq 1$. Property $(\star)$ implies that there exists an L-Lipschitz path $z : [0,1] \to \mathcal{U}(A_\infty \cap A')$ with $z_0 = \mathbf{1}$ and $z_1 = \alpha^{-n}(u_n^*)$.

As α has the (strict) Rokhlin property by assumption, there exists a projection $e \in A_\infty \cap A'$ such that $\mathbf{1} = \sum_{j=0}^{n-1} \alpha^j(e)$. Without loss of generality, we may assume that e also commutes with elements of the form $\alpha^k(z_t)$ as well as $\alpha^k(u)$ for any $k \in \mathbb{Z}$ and $t \in [0,1]$. Set

$$v = \sum_{j=0}^{n-1} \alpha^j(e) \cdot u_j \cdot \alpha^j(z_{j/n}).$$

This defines a unitary in $A_\infty \cap A'$. We compute[12]

$$\begin{aligned}
& v\alpha(v^*) \\
= \;& \sum_{j,k=0}^{n-1} \alpha^j(e) \cdot u_j\alpha^j(z_{j/n}) \cdot \alpha^{k+1}(z_{k/n}^*)\alpha(u_k^*) \cdot \alpha^{k+1}(e) \\
= \;& e \cdot \underbrace{\alpha^n(z_{n-1/n}^*)\alpha(u_{n-1}^*)}_{\approx u_n} + \sum_{j=1}^{n-1} \alpha^j(e) \cdot u_j\alpha^j(\underbrace{z_{j/n}z_{j-1/n}^*}_{\approx \mathbf{1}})\alpha(u_{j-1}^*) \\
\approx_\varepsilon \;& \sum_{j=1}^{n} \alpha^j(e) \cdot \underbrace{u_j\alpha(u_{j-1}^*)}_{=u} \quad = \quad u.
\end{aligned}$$

This finishes the proof. ☐

[12] The key step uses that $z_{j/n}$ is L/n-close to $z_{j-1/n}$ and the fact that we have arranged $L/n \leq \varepsilon$.

3 Evans–Kishimoto intertwining

The following argument crucially uses the approximate cohomology vanishing Lemma 2.4. It is originally due to Evans–Kishimoto; see [7].

Theorem 3.1. *Let A be a separable* C**-algebra with property* ($\star$)*. Let α and β be automorphisms on A with the Rokhlin property. Then α and β are approximately unitarily equivalent if and only if α and β are cocycle conjugate via an approximately inner automorphism on A.*

Idea of proof. Our assumption means that there are unitaries u such that $\mathrm{Ad}(u) \circ \alpha \approx \beta$. If u is close to an α-coboundary, then we may actually get $\beta \approx \mathrm{Ad}(u) \circ \alpha \approx \mathrm{Ad}(v\alpha(v^*)) \circ \alpha = \mathrm{Ad}(v) \circ \alpha \circ \mathrm{Ad}(v^*)$.

In other words, β is point-norm-close to an automorphism which is conjugate to α. But this is still far from the desired statement!

The naive idea would be to choose sequences of u_n, v_n which achieve this approximation better and better as $n \to \infty$. However, why should the inner automorphisms given by v_n approach any given map?

This is where the approximate centrality kicks in: If we first perturb β by a unitary beforehand, we can ensure that α and β are already close in point-norm. Then the map $\mathrm{Ad}(u)$ does only very little, which means that u is approximately central. Our Lemma 2.4 then allows us to pick v as an approximately central unitary. Once we replace α by $\mathrm{Ad}(u) \circ \alpha$, we can repeat this process, but with reversing the roles of α and β.

We will then inductively construct unitaries u_n, v_n in this zigzag fashion (alternating between odd / even n) such that

(i) $\underbrace{\mathrm{Ad}(u_{2(k-1)} u_{2(k-2)} \cdots u_0) \circ \alpha}_{=:\alpha_{2k}} \approx \underbrace{\mathrm{Ad}(u_{2k-1} u_{2k-3} \cdots u_1) \circ \beta}_{=:\beta_{2k+1}}$;

(ii) $u_{2k} \approx v_{2k}\alpha_{2k}(v_{2k}^*)$ and $u_{2k+1} \approx v_{2k+1}\beta_{2k+1}(v_{2k+1}^*)$;

(iii) v_n is approximately central as $n \to \infty$.

Considering $\mathrm{Ad}(v_{2k} \cdots v_0) \circ \alpha \circ \mathrm{Ad}(v_0^* \cdots v_{2k}^*)$ versus $\mathrm{Ad}(v_{2k+1} \cdots v_1) \circ \beta \circ \mathrm{Ad}(v_1^* \cdots v_{2k+1}^*)$, the approximate centrality (iii) ensures that these inner automorphisms converge as $k \to \infty$. The resulting conjugates of α and β will not agree, but condition (i) gives one a unitary sequence correcting the error, which will in turn converge by the coboundary condition (ii). This will result in the desired cocycle conjugacy of α and β.

Detailed proof. Since "$\Leftarrow$" is trivial, we have to show "$\Rightarrow$". So let us assume that α and β are approximately unitarily equivalent.

In what will first be a lot of setup, we are going to apply Lemma 2.4 in a certain zig-zag way to choose unitaries $u_k, v_k \in \mathcal{U}(\mathbf{1}+A)$ and construct new automorphisms $\alpha_{2k}, \beta_{2k+1}$ out of them with certain properties. Once this is done, we will be able to apply a Cauchy sequence argument, and (through a limit process) obtain approximately inner automorphisms φ_0, φ_1 on A and unitaries $w_0, w_1 \in \mathcal{U}(\mathbf{1}+A)$ such that

$$\varphi_0 \circ \mathrm{Ad}(w_0) \circ \alpha \circ \varphi_0^{-1} = \varphi_1 \circ \mathrm{Ad}(w_1) \circ \beta \circ \varphi_1^{-1}.$$

Note: We will use without mention that we may purturb the automorphisms α or β with inner automorphisms without changing the standing assumption that both have the Rokhlin property and that $\alpha \approx_{\mathrm{u}} \beta$.

We will now implement this strategy. Let $\mathcal{F}_n \Subset A$ be an increasing sequence of finite sets in the unit ball whose union is dense.

We set $\alpha_0 = \alpha$ and $\beta_1 = \beta$. Apply Lemma 2.4 to β_1 and choose a pair $(\delta_1, \mathcal{G}_1)$ for the pair $(1/2, \mathcal{F}_1)$. Without loss of generality $\delta_1 \leq 1/2$. Define

$$\mathcal{G}_1' = \beta_1^{-1}(\mathcal{G}_1) \cup \mathcal{F}_1.$$

We may choose a unitary $u_0 \in \mathcal{U}(\mathbf{1}+A)$ such that

$$\max_{a \in \mathcal{G}_1'} \|\beta_1(a) - u_0 \alpha_0(a) u_0^*\| \leq \delta_1/2.$$

Set $\alpha_2 = \mathrm{Ad}(u_0) \circ \alpha_0$, $v_0 = u_0$, and

$$\mathcal{F}_2' = \mathcal{F}_2 \cup v_0 \mathcal{F}_2 v_0^*.$$

Apply Lemma 2.4 to α_2 and choose a pair $(\delta_2, \mathcal{G}_2)$ for the pair $(1/4, \mathcal{F}_2')$ with $\delta_2 \leq \min(1/4, \delta_2)$. Set

$$\mathcal{G}_2' = \alpha_2^{-1}(\mathcal{G}_2) \cup \beta_1^{-1}(\mathcal{G}_1) \cup \mathcal{F}_2.$$

We may choose a unitary $u_1 \in \mathcal{U}(\mathbf{1}+A)$ such that

$$\max_{a \in \mathcal{G}_2'} \|\alpha_2(a) - u_1 \beta_1(a) u_1^*\| \leq \delta_2/2.$$

If we observe closely, we see that in particular

$$\max_{a \in \mathcal{G}_1} \|[u_1, a]\| \leq \delta_1.$$

By our choice of the pair $(\delta_1, \mathcal{G}_1)$, this means that there exists a unitary $v_1 \in \mathcal{U}(\mathbf{1} + A)$ such that

$$\|u_1 - v_1 \beta_1(v_1)^*\| \leq 1/2 \quad \text{and} \quad \max_{a \in \mathcal{F}_1} \|[v_1, a]\| \leq 1/2.$$

We set $\beta_3 = \mathrm{Ad}(u_1) \circ \beta_1$ and

$$\mathcal{F}_3' = \mathcal{F}_3 \cup v_1 \mathcal{F}_3 v_1^*.$$

Apply Lemma 2.4 to β_3 and choose the pair $(\delta_3, \mathcal{G}_3)$ for the pair $(1/8, \mathcal{F}_3')$ with $\delta_3 \leq \min(1/8, \delta_2)$. Set

$$\mathcal{G}_3 = \beta_3^{-1}(\mathcal{G}_3) \cup \alpha_2^{-1}(\mathcal{G}_2) \cup \mathcal{F}_3.$$

Choose a unitary $u_2 \in \mathcal{U}(\mathbf{1} + A)$ such that

$$\max_{a \in \mathcal{G}_3'} \|\beta_3(a) - u_2 \alpha_2(a) u_2\| \leq \delta_3/2.$$

Observing closely again, we see that in particular

$$\max_{a \in \mathcal{G}_2} \|[u_2, a]\| \leq \delta_2.$$

By our choice of the pair $(\delta_2, \mathcal{G}_2)$, this means that there exists a unitary $v_2 \in \mathcal{U}(\mathbf{1} + A)$ such that

$$\|u_2 - v_2 \alpha_2(v_2)^*\| \leq 1/4 \quad \text{and} \quad \max_{a \in \mathcal{F}_2} \|[v_2, a]\| \leq 1/4.$$

Define $\alpha_4 = \mathrm{Ad}(u_2) \circ \alpha_2$ and continue to proceed like above, halving the parameters in each step. Inductively, we obtain unitaries $u_k, v_k \in \mathcal{U}(\mathbf{1} + A)$ and automorphisms $\alpha_{2k}, \beta_{2k+1}$ satisfying the following list of properties:

$$\alpha_{2(k+1)} = \mathrm{Ad}(u_{2k}) \circ \alpha_{2k}; \tag{e3.1}$$

$$\beta_{2k+3} = \mathrm{Ad}(u_{2k+1}) \circ \beta_{2k+1}; \tag{e3.2}$$

$$\max_{a \in \mathcal{F}_{2k+1}} \|\beta_{2k+1}(a) - \alpha_{2k}(a)\| \leq 2^{-(2k+1)}; \tag{e3.3}$$

$$\|u_{2k} - v_{2k}\alpha_{2k}(v_{2k}^*)\| \leq 2^{-2k}; \tag{e3.4}$$

$$\|u_{2k+1} - v_{2k+1}\beta_{2k+1}(v_{2k+1}^*)\| \leq 2^{-(2k+1)}; \tag{e3.5}$$

$$\max_{a\in\mathcal{F}_n'} \|[v_n, a]\| \leq 2^{-n}. \tag{e3.6}$$

$$\mathcal{F}_{2k}' = \mathcal{F}_{2k} \cup \mathrm{Ad}(v_{2(k-1)}\cdots v_0)(F_{2k}); \tag{e3.7}$$

$$\mathcal{F}_{2k+1}' = \mathcal{F}_{2k+1} \cup \mathrm{Ad}(v_{2k-1}\cdots v_1)(F_{2k+1}); \tag{e3.8}$$

For each $n \in \mathbb{N}$, let us define unitaries via

$$U_n = \begin{cases} u_{2k+1}\cdots u_1 & , \quad n = 2k+1 \\ u_{2k}\cdots u_0 & , \quad n = 2k; \end{cases}$$

and

$$V_n = \begin{cases} v_{2k+1}\cdots v_1 & , \quad n = 2k+1 \\ v_{2k}\cdots v_0 & , \quad n = 2k. \end{cases}$$

Taking a close look at conditions (e3.7) and (e3.6) we see that sequences of the form $V_{2k}aV_{2k}^*$ as well as $V_{2k}^*aV_{2k}$ are Cauchy for all $a \in \bigcup_n \mathcal{F}_n$ and hence for all $a \in A$. In particular, the point-norm limit

$$\varphi_0 = \lim_{k\to\infty} \mathrm{Ad}(V_{2k}) \tag{e3.9}$$

exists and yields an (approximately inner) automorphism on A. Similarly, we obtain

$$\varphi_1 = \lim_{k\to\infty} \mathrm{Ad}(V_{2k+1}). \tag{e3.10}$$

Next observe that for the unitaries given by

$$X_n = \begin{cases} V_{2k+1}^* U_{2k+1}\beta(V_{2k+1}) & , \quad n = 2k+1 \\ V_{2k}^* U_{2k}\alpha(V_{2k}) & , \quad n = 2k, \end{cases}$$

conditions (e3.1)+(e3.2) imply

$$\mathrm{Ad}(V_{2k}) \circ \mathrm{Ad}(X_{2k}) \circ \alpha \circ \mathrm{Ad}(V_{2k}^*) = \mathrm{Ad}(U_{2k}) \circ \alpha = \alpha_{2(k+1)} \tag{e3.11}$$

and

$$\mathrm{Ad}(V_{2k+1}) \circ \mathrm{Ad}(X_{2k+1}) \circ \beta \circ \mathrm{Ad}(V_{2k+1}^*) = \mathrm{Ad}(U_{2k+1}) \circ \beta = \beta_{2k+3}. \tag{e3.12}$$

We claim that the unitary sequences X_{2k} and X_{2k+1} are convergent. Indeed, we compute for $k \geq 1$:

$$\begin{array}{rcl} X_{2k} & = & V_{2k}^* U_{2k} \alpha(V_{2k}) \\ & = & V_{2(k-1)}^* v_{2k}^* u_{2k} U_{2(k-1)} \alpha(v_{2k} V_{2(k-1)}) \\ & \stackrel{\text{(e3.1)}}{=} & V_{2(k-1)}^* \underbrace{v_{2k}^* u_{2k} \alpha_{2k}(v_{2k}}_{\approx \mathbf{1}} V_{2(k-1)}) U_{2(k-1)} \\ & \stackrel{\text{(e3.4)}}{\approx_{2^{-2k}}} & V_{2(k-1)}^* \alpha_{2k}(V_{2(k-1)}) U_{2(k-1)} \\ & \stackrel{\text{(e3.1)}}{=} & V_{2(k-1)}^* U_{2(k-1)} \alpha(V_{2(k-1)}) \;=\; X_{2(k-1)}. \end{array}$$

In particular, the unitaries X_{2k} form a Cauchy sequence, and therefore have a limit

$$w_0 = \lim_{k \to \infty} X_{2k} \;\in\; \mathcal{U}(\mathbf{1} + A). \tag{e3.13}$$

Similarly we obtain

$$w_1 = \lim_{k \to \infty} X_{2k+1} \;\in\; \mathcal{U}(\mathbf{1} + A). \tag{e3.14}$$

We claim that these now do the trick as claimed earlier. Indeed, combining everything we have done so far, we get

$$\begin{array}{ll} \varphi_0 \circ \mathrm{Ad}(w_0) \circ \alpha \circ \varphi_0^{-1} & \\ \stackrel{\text{(e3.9),(e3.13)}}{=} & \lim\limits_{k \to \infty} \mathrm{Ad}(V_{2k}) \circ \mathrm{Ad}(X_{2k}) \circ \alpha \circ \mathrm{Ad}(V_{2k}^*) \\ \stackrel{\text{(e3.11)}}{=} & \lim\limits_{k \to \infty} \alpha_{2(k+1)} \\ \stackrel{\text{(e3.3)}}{=} & \lim\limits_{k \to \infty} \beta_{2k+3} \\ \stackrel{\text{(e3.12)}}{=} & \lim\limits_{k \to \infty} \mathrm{Ad}(V_{2k+1}) \circ \mathrm{Ad}(X_{2k+1}) \circ \beta \circ \mathrm{Ad}(V_{2k+1}^*) \\ \stackrel{\text{(e3.10),(e3.14)}}{=} & \varphi_1 \circ \mathrm{Ad}(w_1) \circ \beta \circ \varphi_1^{-1}. \end{array}$$

This finishes the proof. □

If we restrict to some special classes of C*-algebras satisfying property ($\star$) and appeal to classification theory, we can obtain the following corollaries.

Corollary 3.2 (Evans–Kishimoto [7]). *Let A be an AF C*-algebra. Let α and β be automorphisms on A with the Rokhlin property. Then α and β are cocycle conjugate if and only if the induced automorphisms $K_0(\alpha)$ and $K_0(\beta)$ on the scaled ordered K_0-group of A are conjugate.*

Proof. Let κ be an automorphism on the scaled ordered K_0-group such that $K_0(\alpha)\circ\kappa = \kappa\circ K_0(\beta)$. By the classification theory of AF algebras [**?**], there is an automorphism σ on A such that $K_0(\sigma) = \kappa$. If we replace β by $\sigma\circ\beta\circ\sigma^{-1}$, we may thus assume that $K_0(\alpha) = K_0(\beta)$. Appealing again to classification theory, this means that α and β are approximately unitarily equivalent. As AF algebras have property $(\star)$, the claim follows from Theorem 3.1. □

Corollary 3.3. *Every infinite-dimensional UHF algebra carries a unique automorphism with the Rokhlin property up to cocycle conjugacy.*

Although surely known to some experts, the next corollary has never been recorded anywhere. It is an easy consequence of Theorem 3.1 but goes beyond what can be found in the literature. Although the underlying class of examples is clearly very different from the AF case, I would like to draw the reader's attention to the fact that the proof is still virtually the same as for AF algebras.

Corollary 3.4. *Let A be a separable, nuclear* C**-algebra with $A \cong A\otimes\mathcal{O}_2$. Assume that A is either unital or stable. Let α and β be automorphisms on A with the Rokhlin property. Then α and β are cocycle conjugate if and only if their induced homeomorphisms on the prime ideal space* Prim(A) *are conjugate.*

Proof. We proceed similarly as above. Let κ be a homeomorphism on the prime ideal space so that $\alpha(\kappa(J)) = \kappa(\beta(J))$ for all prime ideals J in A. Appealing to the classification of nuclear $\mathcal{O}_2$-absorbing C*-algebras [8], there exists an automorphism σ on A that lifts κ. Replacing β by $\sigma\circ\beta\circ\sigma^{-1}$, we may assume that α and β induce the same maps on the ideal lattice. But this implies that they are approximately unitarily equivalent. As A has property $(\star)$, the claim follows from Theorem 3.1. □

4 Closing remarks

Remark 4.1. Property $(\star)$ from Notation 2.1 has a useful generalization in the non-unital case:

Instead of requiring the path to be L-Lipschitz and to have the given unitary u as the endpoint, we could require instead that the path is approximately L-Lipschitz in the strict topology[13] and that its endpoint approx-

[13]That is, maps of the form $[t\mapsto u_t\cdot a]$ are L-Lipschitz for contractions $a\in\mathcal{F}$

imates u in the strict topology. In sequence algebra language, this condition would mean that the image of the unitary group $\mathcal{U}(\mathbf{1} + A_\infty \cap A')$ in $F_\infty(A) = (A_\infty \cap A')/(A_\infty \cap A^\perp)$ is connected.

This weaker property can actually be used to obtain all the relevant results we have obtained so far, with some minor modifications in some statements and proofs. This is for example relevant when handling certain stably projectionless C*-algebras such as the Razak–Jacelon algebra $\mathcal{W}$ or other C*-algebras absorbing it; see [14, 6], [33, Section 7] and [37, Theorem 5.12]. Such C*-algebras turn out to satisfy this weaker property, but in general not our property ($\star$) from Notation 2.1.

Remark 4.2. One conceptual reason why Theorem 3.1 has any right to work is that, under property ($\star$), approximate unitary equivalence of automorphisms coincides with asymptotic unitary equivalence.[14] When we have a C*-algebra A for which this phenomenon fails, we quickly reach the limits of our methods so far: It is clear that certain properties of automorphisms such as approximate or asymptotic innerness are preserved under cocycle conjugacy. So if A carries two approximately inner automorphisms α and β with the Rokhlin property and only one of them is asymptotically inner, we see that the statement of Theorem 3.1 cannot hold.

Example 4.3. Let $\theta \in [0,1] \setminus \mathbb{Q}$ and consider the irrational rotation algebra

$$A_\theta = \mathrm{C}^*\big(u, v \text{ unitaries } \mid uv = e^{2\pi i\theta} vu\big).$$

Let $\rho \in \mathbb{R} \setminus \mathbb{Q}[\theta]$ and consider the automorphism α on A_θ defined by $\alpha(u) = e^{2\pi i \rho} u$ and $\alpha(v) = v$. Then α is approximately inner, has the Rokhlin property, but is not asymptotically inner.[15] On the other hand, if we fix some isomorphism $A_\theta \cong A_\theta \otimes \mathcal{Z}$ and induce an automorphism β on A_θ via pulling back $\mathrm{id} \otimes \sigma$ for some sufficiently outer automorphism σ on $\mathcal{Z}$, it is possible obtain an asymptotically inner automorphism with the Rokhlin property.[16]

Remark 4.4. If we want to go beyond C*-algebras satisfying the (very restrictive) property ($\star$), we see from the above that we have to take into

[14] Coming up with the (elementary) proof of this fact is a good exercise.

[15] The relevant obstruction, the so-called rotation map, does not vanish.

[16] The precise details are beyond the scope of these notes. However, as irrational rotation algebras are approximately divisible, one could alternatively apply the generalized construction from the proof of Example 1.5.

account asymptotic unitary equivalence of automorphisms. In addition, we can see from the proof of Lemma 2.4 that it was crucal for the cohomology vanishing argument to have some method of connecting approximately central unitaries to the unit in an approximately central way, and moreover we need to be able to do this with a uniform Lipschitz constant.

Without property ($\star$) and in particular when $K_1 \neq 0$, this is of course impossible in general. The correct substitute for property ($\star$) turns out to be the so-called *basic homotopy lemma* in its various forms, which usually tells us for which kind of unitaries one may find such homotopies; see for example [1, 22, 23, 9]. This requires one to deal with most of the C*-algebraic secondary invariants, which are already essential in the proof of the Elliott classification theorems, even if they never show up in the final statements.

All of this showcases the additional layers of technical difficulty that need to be overcome to obtain more general classification results for single automorphisms, let alone actions of more general groups; see for example [20, 28, 24].

Remark 4.5. In the unital case, the Rokhlin property requires the existence of projections. This puts another obvious limitation on our methods so far: If the unital C*-algebra A has no non-trivial projections, it cannot have automorphisms with the Rokhlin property. It is therefore a challenge to figure out what to do in the absence of projections, such as for the Jiang–Su algebra $A = \mathcal{Z}$.

One promising approach appears to be Matui–Sato weak Rokhlin property [30, 31], which is formulated purely in terms of positive elements. It then becomes a non-trivial fact that, within the relevant interesting cases, their weak Rokhlin property for an automorphism $\alpha \in \mathrm{Aut}(A)$ is equivalent to saying that some (or all) UHF stabilization $\alpha \otimes \mathrm{id}_{\mathbb{U}} \in \mathrm{Aut}(A \otimes \mathbb{U})$ has the regular Rokhlin property.

References

[1] O. Bratteli, G. A. Elliott, D. Evans, A. Kishimoto: Homotopy of a pair of approximately commuting unitaries in a simple C*-algebra. J. Funct. Anal. 160 (1998), no. 2, pp. 466–523.

[2] A. Connes: Une classification des facteurs de type III. Ann. Sci. École Norm. Sup. 6 (1973), pp. 133–252.

[3] A. Connes: Outer conjugacy classes of automorphisms of factors. Ann. Sci. École Norm. Sup. 4 (1975), no. 8, pp. 383–419.

[4] A. Connes: Classification of injective factors. Cases II_1, II_∞, III_λ, $\lambda \neq 1$. Ann. Math. 74 (1976), pp. 73–115.

[5] A. Connes: Periodic automorphisms of the hyperfinite factors of type II_1. Acta Sci. Math. 39 (1977), pp. 39–66.

[6] G. A. Elliott, G. Gong, H. Lin, Z. Niu: The classification of simple separable KK-contractible C*-algebras with finite nuclear dimension (2017). URL `https://arxiv.org/abs/1712.09463`.

[7] D. Evans, A. Kishimoto: Trace scaling automorphisms of certain stable AF algebras. Hokkaido Math. J. 26 (1997), pp. 211–224.

[8] J. Gabe: A new proof of Kirchberg's $\mathcal{O}_2$-stable classification (2017). URL `https://arxiv.org/abs/1706.03690`.

[9] G. Gong, H. Lin, Z. Niu: Classification of simple amenable $\mathcal{Z}$-stable C*-algebras (2015). URL `http://arxiv.org/abs/1501.00135`.

[10] U. Haagerup, M. Rørdam: Perturbation of the rotation C*-algebras and of the Heisenberg commutation relation. Duke Math. J. 77 (1995), pp. 627–656.

[11] R. H. Herman, A. Ocneanu: Stability for integer actions on UHF C*-algebras. J. Funct. Anal. 59 (1984), pp. 132–144.

[12] M. Izumi: Finite group actions on C*-algebras with the Rohlin property I. Duke Math. J. 122 (2004), no. 2, pp. 233–280.

[13] M. Izumi, H. Matui: $\mathbb{Z}^2$-actions on Kirchberg algebras. Adv. Math. 224 (2010), pp. 355–400.

[14] B. Jacelon: A simple, monotracial, stably projectionless C*-algebra. J. Lond. Math. Soc. 87 (2013), no. 2, pp. 365–383.

[15] V. F. R. Jones: Actions of finite groups on the hyperfinite type II_1 factor. Mem. Amer. Math. Soc. 28 (1980), no. 237.

[16] Y. Katayama, C. E. Sutherland, M. Takesaki: The characteristic square of a factor and the cocycle conjugacy of discrete group actions on factors. Invent. Math. 132 (1998), pp. 331–380.

[17] T. Katsura, H. Matui: Classification of uniformly outer actions of $\mathbb{Z}^2$ on UHF algebras. Adv. Math 218 (2008), pp. 940–968.

[18] Y. Kawahigashi, C. Sutherland, M. Takesaki: The structure of the automorphism group of an injective factor and the cocycle conjugacy of discrete abelian group actions. Acta Math. 169 (1992), pp. 105–130.

[19] A. Kishimoto: The Rohlin property for automorphisms of UHF algebras. J. reine angew. Math. 465 (1995), pp. 183–196.

[20] A. Kishimoto: Automorphisms of AT algebras with the Rohlin property. J. Operator Theory 40 (1998), pp. 277–294.

[21] A. Kishimoto: Unbounded derivations in AT algebras. J. Funct. Anal. 160 (1998), pp. 270–311.

[22] A. Kishimoto, A. Kumjian: The Ext class of an approximately inner automorphism, II. J. Operator Theory 46 (2001), pp. 99–122.

[23] H. Lin: Approximate homotopy of homomorphisms from C(X) into a simple C*-algebra. Mem. Amer. Math. Soc. 205 (2010), no. 963, pp. 1–131.

[24] H. Lin: Kishimoto's conjugacy theorems in simple C*-algebras of tracial rank one. Integr. Equ. Oper. Theory 83 (2015), no. 1, pp. 95–150.

[25] T. Masuda: Evans-Kishimoto type argument for actions of discrete amenable groups on McDuff factors. Math. Scand. 101 (2007), pp. 48–64.

[26] T. Masuda: Unified approach to the classification of actions of discrete amenable groups on injective factors. J. reine angew. Math. 683 (2013), pp. 1–47.

[27] H. Matui: Classification of outer actions of $\mathbb{Z}^N$ on $\mathcal{O}_2$. Adv. Math. 217 (2008), pp. 2872–2896.

[28] H. Matui: $\mathbb{Z}$-actions on AH algebras and $\mathbb{Z}^2$-actions on AF algebras. Comm. Math. Phys. 297 (2010), pp. 529–551.

[29] H. Matui: $\mathbb{Z}^N$-actions on UHF algebras of infinite type. J. reine angew. Math 657 (2011), pp. 225–244.

[30] H. Matui, Y. Sato: $\mathcal{Z}$-stability of crossed products by strongly outer actions. Comm. Math. Phys. 314 (2012), no. 1, pp. 193–228.

[31] H. Matui, Y. Sato: $\mathcal{Z}$-stability of crossed products by strongly outer actions II. Amer. J. Math. 136 (2014), pp. 1441–1497.

[32] H. Nakamura: Aperiodic automorphisms of nuclear purely infinite simple C*-algebras. Ergod. Th. Dynam. Sys. 20 (2000), pp. 1749–1765.

[33] N. Nawata: Trace scaling automorphisms on $\mathcal{W}\otimes\mathbb{K}$ (2017). URL `https://arxiv.org/abs/1704.02414`.

[34] A. Ocneanu: Actions of discrete amenable groups on von Neumann algebras, volume 1138. Springer-Verlag, Berlin (1985).

[35] Y. Sato: The Rohlin property for automorphisms of the Jiang-Su algebra. J. Funct. Anal. 259 (2010), no. 2, pp. 453–476.

[36] C. Sutherland, M. Takesaki: Actions of discrete amenable groups on injective factors of type III_λ, $\lambda \neq 1$. Pacific J. Math. 137 (1989), pp. 405–444.

[37] G. Szabó: The classification of Rokhlin flows on C*-algebras (2017). URL `https://arxiv.org/abs/1706.09276`.