# A peek into the classification of $\mathrm{C}^*$-dynamics

## UK Virtual Operator Algebras Seminar

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**Objects of interest:** $\mathrm{C}^*$-dynamical systems $(A, \alpha, G)$, where

- $A$ is a $\mathrm{C}^*$-algebra
- $G$ is a locally compact group
- $\alpha : G \curvearrowright A$ is a continuous action.

**Overarching goal:** Exploit invariants to classify up to cocycle conjugacy.

## Definition

Let $\alpha : G \curvearrowright A$ be an action.

- An $\alpha$-cocycle is a strictly continuous map $\mathbb{u} : G \to \mathcal{U}(\mathcal{M}(A))$ with $\mathbb{u}_{gh} = \mathbb{u}_g \alpha_g(\mathbb{u}_h)$ for all $g, h \in G$. In this case, $\alpha^{\mathbb{u}}_\bullet := \mathrm{Ad}(\mathbb{u}_\bullet) \circ \alpha_\bullet$ is another action.
- $\alpha$ is said to be (cocycle) conjugate to $\beta : G \curvearrowright B$, if there is an isomorphism $\varphi : A \to B$ (and an $\alpha$-cocycle $\mathbb{u}$) such that
$$\alpha^{\mathbb{u}}_g = \varphi^{-1} \circ \beta_g \circ \varphi, \quad g \in G.$$

**In this talk** $\mathrm{C}^*$-algebras shall be unital and groups discrete. (convenience)

The "overarching goal" is meant as an extension of the Elliott program, i.e., the Elliott program should correspond to $G = \{1\}$.

**In particular** $A$ is often simple amenable $\mathcal{Z}$-stable...

In order to introduce you to this subject, I would like to preview the important slogan (or meta-idea) for today.

> When classifying a class of $\mathrm{C}^*$-dynamics, first understand how to classify the underlying $\mathrm{C}^*$-algebras. Then find a way to reduce *equivariant classification* to *non-equivariant classification* by means of an *averaging process* that exploits *amenability*.

In a bit, we will discuss the classification of finite group actions with the Rokhlin property, where this theme can be nicely demonstrated with not too involved arguments.

Before looking at $\mathrm{C}^*$-dynamics, first we need to go through some basics.

### Theorem (Elliott intertwining)

*Let $A$ and $B$ be two separable $\mathrm{C}^*$-algebras. Suppose there are $*$-homomorphisms $\varphi: A \to B$ and $\psi: B \to A$ with $\psi \circ \varphi \approx_{\mathrm{u}} \mathrm{id}_A$ and $\varphi \circ \psi \approx_{\mathrm{u}} \mathrm{id}_B$. Then $\varphi$ and $\psi$ are approximately unitarily equivalent to mutually inverse isomorphisms.*

Fortunately for us there is an easy dynamical analog when $G$ is finite.

### Definition

Let two actions $\alpha: G \curvearrowright A$ and $\beta: G \curvearrowright B$ be given. Two equivariant $*$-homomorphisms $\varphi, \psi: (A, \alpha) \to (B, \beta)$ are approximately $G$-unitarily equivalent, $\varphi \approx_{\mathrm{u},G} \psi$, if we find unitaries $v_n \in \mathcal{U}(B^\beta)$ such that $\psi = \lim_{n\to\infty} \mathrm{Ad}(v_n) \circ \varphi$.

## Definition (repeated)

Let two actions $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be given. Two equivariant $*$-homomorphisms $\varphi, \psi : (A, \alpha) \to (B, \beta)$ are approximately $G$-unitarily equivalent, $\varphi \approx_{\mathrm{u},G} \psi$, if we find unitaries $v_n \in \mathcal{U}(B^\beta)$ such that $\psi = \lim_{n\to\infty} \mathrm{Ad}(v_n) \circ \varphi$.

By copying the non-dynamical proof almost verbatim, one gets:

## Theorem (dynamical Elliott intertwining for finite groups)

*Let $G$ be a finite group, and let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be two actions on separable $\mathrm{C}^*$-algebras. Suppose there are equivariant $*$-homomorphisms $\varphi : (A, \alpha) \to (B, \beta)$ and $\psi : (B, \beta) \to (A, \alpha)$ with $\psi \circ \varphi \approx_{\mathrm{u},G} \mathrm{id}_A$ and $\varphi \circ \psi \approx_{\mathrm{u},G} \mathrm{id}_B$. Then $\varphi$ and $\psi$ are approximately $G$-unitarily equivalent to mutually inverse conjugacies.*

For the applications yet to come, here's a big black box regarding the so-called existence/uniqueness theorems underpinning the modern Elliott program. Let $\mathfrak{E}$ denote the class of separable unital simple amenable $\mathcal{Z}$-stable $\mathrm{C}^*$-algebras satisfying the UCT.

## Theorem (many hands)

*Let $A, B \in \mathfrak{E}$ and let $\varphi, \psi : A \to B$ be two unital $*$-homomorphisms. Then $\varphi \approx_{\mathrm{u}} \psi$ if and only if $\underline{\mathrm{Ell}}(\varphi) = \underline{\mathrm{Ell}}(\psi)$.*[1]

## Theorem (many hands)

*Let $A, B \in \mathfrak{E}$. For any morphism $\zeta : \underline{\mathrm{Ell}}(A) \to \underline{\mathrm{Ell}}(B)$, there exists a unital $*$-homomorphism $\varphi : A \to B$ with $\underline{\mathrm{Ell}}(\varphi) = \zeta$.*

[1] On this slide "$\underline{\mathrm{Ell}}$" denotes the *total Elliott invariant*, which includes traces and total/algebraic K-theory. It is finer than the ordinary Elliott invariant, but contains the same information about isomorphism classes.

Now let us finally look at the classification of Rokhlin actions!

## Definition (Izumi)

Let $G$ be a finite group and $A$ a separable unital $\mathrm{C}^*$-algebra. An action $\alpha : G \curvearrowright A$ is said to have the Rokhlin property, if there exists a sequence of projections $e_n \in A$ such that

- $\|[a, e_n]\| \to 0$ for all $a \in A$
- $\sum_{g\in G} \alpha_g(e_n) \to \mathbf{1}_A$.

$(A, \alpha) \approx (A \otimes \mathcal{C}(G), \alpha \otimes \text{shift})$

Although there exist plenty of example of such actions, the Rokhlin property is quite restrictive. However, as shown in the work of Izumi, Rokhlin actions can be very effectively classified.

## Example (Prototypical one)

Let $G$ be a finite group with its left-regular representation $\lambda : G \to \mathcal{B}(\ell^2(G)) = M_{|G|}$. Then $\gamma = \mathrm{Ad}(\lambda)^{\otimes\infty} : G \curvearrowright M_{|G|^\infty}$ has the Rokhlin property.

Going forward, I wish to convince you that for Rokhlin actions, the previous existence/uniqueness theorems imply their own equivariant versions, which will ultimately give us equivariant classification.

We shall start with the following reduction principle regarding the uniqueness of $*$-homomorphisms.

**Theorem**

*Let $G$ be a finite group. Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions on separable unital $\mathrm{C}^*$-algebras, and assume $\beta$ has the Rokhlin property. For two unital equivariant $*$-homomorphisms $\varphi, \psi : (A, \alpha) \to (B, \beta)$, we have $\varphi \approx_{\mathrm{u},G} \psi$ if and only if $\varphi \approx_{\mathrm{u}} \psi$.*

## Theorem (continued)

*Let $G$ be a finite group. Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions on separable unital $\mathrm{C}^*$-algebras, and assume $\beta$ has the Rokhlin property. For two unital equivariant $*$-homomorphisms $\varphi, \psi : (A, \alpha) \to (B, \beta)$, we have $\varphi \approx_{\mathrm{u},G} \psi$ if and only if $\varphi \approx_{\mathrm{u}} \psi$.*

**Sketch of proof:** Suppose $v_n \in \mathcal{U}(B)$ satisfies $\psi = \lim_{n\to\infty} \mathrm{Ad}(v_n) \circ \varphi$. Note that since $\varphi$ and $\psi$ were equivariant, one also has

$$\lim_{n\to\infty} \mathrm{Ad}(\beta_g(v_n)) \circ \varphi = \lim_{n\to\infty} \beta_g \circ \mathrm{Ad}(v_n) \circ \varphi \circ \alpha_g^{-1} = \beta_g \circ \psi \circ \alpha_g^{-1} = \psi.$$

Let $e_n \in B$ be a sequence of projections as required by the Rokhlin property. Without loss of generality we may assume $\|[e_n, v_n]\| \to 0$.

Then we find a sequence of unitaries $\mathcal{U}(B^\beta) \ni u_n \approx \sum_{g\in G} \beta_g(e_n v_n)$.

Then:
$$\mathrm{Ad}(u_n) \circ \varphi \approx \sum_{g\in G} \beta_g(e_n) \cdot \underbrace{\mathrm{Ad}(\beta_g(v_n)) \circ \varphi}_{\approx \psi} \approx \psi.$$

Thus the sequence $u_n$ witnesses $\varphi \approx_{\mathrm{u},G} \psi$. □

Next we discuss the reduction principle regarding existence.

### Theorem (Gardella–Santiago)

*Let $G$ be a finite group. Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions on separable unital $\mathrm{C}^*$-algebras, and assume $\beta$ has the Rokhlin property. Suppose $\varphi : A \to B$ is a unital $*$-homomorphism with $\varphi \circ \alpha_g \approx_{\mathrm{u}} \beta_g \circ \varphi$ for all $g \in G$. Then there exists a unital equivariant $*$-homomorphism $\psi : (A, \alpha) \to (B, \beta)$ with $\varphi \approx_{\mathrm{u}} \psi$.*

**Sketch of proof:** For each $h \in G$ let $w_h \in \mathcal{U}(B)$ be some unitaries such that

$$\beta_h \circ \varphi \approx \mathrm{Ad}(w_h) \circ \varphi \circ \alpha_h, \quad h \in G.$$

Let $e \in B$ be a *good enough* projection as required by the Rokhlin property. Then we find a unitary $\mathcal{U}(B) \ni v \approx \sum_{h \in G} \beta_h(e) w_h$.

Set $\varphi_1 = \mathrm{Ad}(v) \circ \varphi$.

**Sketch of proof: (continued)**

We find a unitary $\mathcal{U}(B) \ni v \approx \sum_{h\in G} \beta_h(e)w_h$ and set $\varphi_1 = \mathrm{Ad}(v)\circ\varphi$.

We observe for all $g \in G$:

$$\begin{aligned}
\beta_g \circ \varphi_1 &\approx \textstyle\sum_{h\in G} \beta_{gh}(e)\cdot \beta_g \circ \underbrace{\mathrm{Ad}(w_h)\circ\varphi}_{\approx \beta_h\circ\varphi\circ\alpha_h^{-1}} \\
&\approx \textstyle\sum_{h\in G} \beta_{gh}(e)\cdot \beta_{gh}\circ\varphi\circ\alpha_h^{-1} \\
&= \textstyle\sum_{h\in G} \beta_h(e)\cdot \underbrace{\beta_h\circ\varphi\circ\alpha_h^{-1}}_{\approx \mathrm{Ad}(w_h)\circ\varphi} \circ\alpha_g \\
&\approx \varphi_1\circ\alpha_g.
\end{aligned}$$

Repeat this inductively and get a sequence of maps $\varphi_1, \varphi_2, \varphi_3, \dots$ for which these approximations holds better and better. If one does this carefully, one can arrange the maps $(\varphi_n)$ to be Cauchy in point-norm, which allows us to get the desired map as $\psi = \lim_{n\to\infty} \varphi_n$. □

As a consequence of all of this, we get the following classification result:

**Theorem**

*Let $G$ be a finite group. Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be two Rokhlin actions on classifiable $\mathrm{C}^*$-algebras. Then $\alpha$ and $\beta$ are conjugate if and only if*

$$\underline{\mathrm{Ell}}(\alpha) : G \curvearrowright \underline{\mathrm{Ell}}(A) \quad \textit{and} \quad \underline{\mathrm{Ell}}(\beta) : G \curvearrowright \underline{\mathrm{Ell}}(B)$$

*are conjugate.*

**Proof:** Assume that $\zeta : \underline{\mathrm{Ell}}(A) \to \underline{\mathrm{Ell}}(B)$ is an equivariant isomorphism. By the black box, we find $*$-homomorphisms $\varphi_0 : A \to B$ and $\psi_0 : B \to A$ lifting $\zeta$ and $\zeta^{-1}$, respectively. Since $\zeta$ is equivariant, it follows from the black box that these maps are equivariant modulo $\approx_{\mathrm{u}}$. By the reduction trick, we may find equivariant $*$-homomorphisms $\varphi : (A, \alpha) \to (B, \beta)$ and $\psi : (B, \beta) \to (A, \alpha)$ lifting $\zeta$ and $\zeta^{-1}$. Using again the black box and the other reduction trick, we see $\psi \circ \varphi \approx_{\mathrm{u},G} \mathrm{id}_A$ and $\varphi \circ \psi \approx_{\mathrm{u},G} \mathrm{id}_B$. The dynamical Elliott intertwining takes care of the rest.

With a bit more work one can actually obtain a more satisfactory version of this result, but this involves pure homological algebra.

**Theorem (Izumi; published only in part)**

*Let $G$ be a finite group. Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be two Rokhlin actions on classifiable $\mathrm{C}^*$-algebras. Then $\alpha$ and $\beta$ are conjugate if and only if*

$$\mathrm{Ell}(\alpha) : G \curvearrowright \mathrm{Ell}(A) \quad \textit{and} \quad \mathrm{Ell}(\beta) : G \curvearrowright \mathrm{Ell}(B)$$

*are conjugate.*

**Example**

For any finite group $G$, there is a unique Rokhlin action $G \curvearrowright \mathcal{O}_2$. For example, the two actions

$$\alpha : \mathbb{Z}_2 \curvearrowright \mathcal{O}_2 = \mathrm{C}^*(s_1, s_2), \quad \alpha(s_j) = (-1)^j s_j$$

and

$$\beta : \mathbb{Z}_2 \curvearrowright \mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2, \quad \beta(x_1 \otimes x_2) = x_2 \otimes x_1$$

are conjugate.

I would like to leave you with some remarks about the classification of more general $\mathrm{C}^*$-dynamics:

- I started the presentation talking about *cocycle* conjugacy, after which no cocycles were to be seen. It just so happens that cocycles can always be trivialized for Rokhlin actions, which is special.
- But don't be fooled, the cocycles are **very** important in general. As soon as $G$ is infinite, classification up to conjugacy is not feasible.
- In general, working with genuine equivariant maps can be too restrictive. My suggested approach is to work in a category where an arrow between $\mathrm{C}^*$-dynamical systems is a pair
$$(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta),$$
where $\mathbb{u}$ is a $\beta$-cocycle and $\varphi$ is a $*$-homomorphism which is equivariant with respect to $\alpha$ and $\beta^{\mathbb{u}}$.
- Another warning: The theory of Rokhlin actions might lead you to believe that ultimately, nice actions on classifiable $\mathrm{C}^*$-algebras are determined by how they act on the Elliott invariant. Although the analogous statement is true for injective factors, this expectation fails quite spectacularly in the general $\mathrm{C}^*$-context.

**Thank you for your attention!**