An introduction to C*-algebras

Workshop Model Theory and Operator Algebras
BIRS, Banff

Gábor Szabó
KU Leuven
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We will denote by $\mathcal{H}$ a complex Hilbert space with inner product $\langle \cdot | \cdot \rangle$, and $\mathcal{B}(\mathcal{H})$ the set of all bounded linear operators $\mathcal{H} \to \mathcal{H}$. It becomes a Banach algebra with the operator norm.
We will denote by \( \mathcal{H} \) a complex Hilbert space with inner product \( \langle \cdot | \cdot \rangle \), and \( \mathcal{B}(\mathcal{H}) \) the set of all bounded linear operators \( \mathcal{H} \to \mathcal{H} \). It becomes a Banach algebra with the operator norm.

**Recall**

For \( a \in \mathcal{B}(\mathcal{H}) \), the adjoint operator \( a^* \in \mathcal{B}(\mathcal{H}) \) is the unique operator satisfying the formula

\[
\langle a\xi_1 | \xi_2 \rangle = \langle \xi_1 | a^*\xi_2 \rangle, \quad \xi_1, \xi_2 \in \mathcal{H}.
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Then the adjoint operation \( a \mapsto a^* \) is an involution, i.e., it is anti-linear and satisfies \((ab)^* = b^*a^*\).
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$$\langle a\xi_1 | \xi_2 \rangle = \langle \xi_1 | a^* \xi_2 \rangle, \quad \xi_1, \xi_2 \in \mathcal{H}.$$ 

Then the adjoint operation $a \mapsto a^*$ is an involution, i.e., it is anti-linear and satisfies $(ab)^* = b^*a^*$.

**Observation**

One always has $\|a^*a\| = \|a\|^2$.

**Proof:** Since $\|a^*\| = \|a\|$ is rather immediate from the definition, “$\leq$” is clear. For “$\geq$”, observe

$$\|a\xi\|^2 = \langle a\xi | a\xi \rangle = \langle \xi | a^*a\xi \rangle \leq \|a^*a\xi\|, \quad \|\xi\| = 1.$$
Definition

An (abstract) $\mathbb{C}^*$-algebra is a complex Banach algebra $A$ with an involution $a \mapsto a^*$ satisfying the $\mathbb{C}^*$-identity

$$\|a^*a\| = \|a\|^2, \quad a \in A.$$  

We say $A$ is unital, if there exists a unit element $1 \in A$. 
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**Definition**

A concrete $\mathbb{C}^*$-algebra is a self-adjoint subalgebra $A \subseteq \mathcal{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$, which is closed in the operator norm.
An (abstract) C*-algebra is a complex Banach algebra $A$ with an involution $a \mapsto a^*$ satisfying the C*-identity

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A concrete C*-algebra is a self-adjoint subalgebra $A \subseteq \mathcal{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$, which is closed in the operator norm.

As the operator norm satisfies the C*-identity, every concrete C*-algebra is an abstract C*-algebra.
Example

For some compact Hausdorff space $X$, we may consider

$$\mathcal{C}(X) = \{\text{continuous functions } X \to \mathbb{C}\}.$$  

With pointwise addition and multiplication, $\mathcal{C}(X)$ becomes a **commutative** abstract C*-algebra if we equip it with the adjoint operation

$$f^*(x) = \overline{f(x)}$$

and the norm

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$
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Fact (Spectral theory)

*As an abstract $\mathbb{C}^*$-algebra, $\mathcal{C}(X)$ remembers $X$.*
The **goal for this lecture** is to go over the spectral theory of Banach algebras and $C^*$-algebras, culminating in:

**Theorem (Gelfand–Naimark)**

*Every (unital) commutative $C^*$-algebra is isomorphic to $C(X)$ for some compact Hausdorff space $X$.*
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The **goal for the next lecture** is to showcase some applications, and discuss the GNS construction, in particular:

**Theorem (Gelfand–Naimark–Segal)**

*Every abstract $\mathbb{C}^*$-algebra can be expressed as a concrete $\mathbb{C}^*$-algebra.*
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The **goal for the next lecture** is to showcase some applications, and discuss the GNS construction, in particular:

**Theorem (Gelfand–Naimark–Segal)**

*Every abstract $C^*$-algebra can be expressed as a concrete $C^*$-algebra.***

The **goal for tomorrow** is to cover examples and advanced topics.
From now on, we will assume that $A$ is a Banach algebra with unit. We identify $\mathbb{C} \subseteq A$ as $\lambda \mapsto \lambda \cdot 1$.

**Observation (Neumann series)**

If $x \in A$ with $\|1 - x\| < 1$, then $x$ is invertible. In fact

$$x^{-1} = \sum_{n=0}^{\infty} (1 - x)^n.$$
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**Proof:**

$$x \sum_{n=0}^{\infty} (1 - x)^n = \sum_{n=0}^{\infty} ((1 - x)^n - (1 - x)^{n+1})) = 1.$$
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**Observation**

The set of invertibles in $A$ is open.

**Proof:** If $z$ is invertible and $x$ is any element with $\|z - x\| < \|z^{-1}\|^{-1}$, then $\|1 - z^{-1}x\| < 1$. By the above $z^{-1}x$ is invertible, but then $x$ is also invertible.
Definition

For an element $x \in A$, its spectrum is defined as

$$\sigma(x) = \{ \lambda \in \mathbb{C} \mid \lambda - x \text{ is not invertible in } A \} \subseteq \mathbb{C}.$$
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Elements in the spectrum may be seen as generalized eigenvalues of an operator.

Observation

The spectrum \( \sigma(x) \) is a compact subset of \( \{ \lambda \mid |\lambda| \leq \|x\| \} \). One defines the spectral radius of \( x \) as \( r(x) = \max_{\lambda \in \sigma(x)} |\lambda| \leq \|x\| \).
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The spectrum $\sigma(x)$ is a compact subset of $\{ \lambda \mid |\lambda| \leq \|x\| \}$. One defines the spectral radius of $x$ as $r(x) = \max_{\lambda \in \sigma(x)} |\lambda| \leq \|x\|$.

**Theorem**

The spectrum $\sigma(x)$ of every element $x \in A$ is non-empty.

(The proof involves a non-trivial application of complex analysis.)
Definition

A character on $A$ is a non-zero multiplicative linear functional $A \to \mathbb{C}$. 

Observation

A character $\phi : A \to \mathbb{C}$ is automatically continuous, in fact $\|\phi\| = 1$.

Proof:

As $\phi$ is non-zero, we have $0 \neq \phi(1) = \phi(1)^2$, hence $\phi(1) = 1$.

If $x$ were to satisfy $|\phi(x)| > \|x\|$, then $\phi(x) - x$ is invertible by the Neumann series trick. However, it lies in the kernel of $\phi$, which yields a contradiction.

Definition

For commutative $A$, we define its spectrum (aka character space) as $\hat{A} = \{\text{characters } \phi : A \to \mathbb{C}\}$.

Due to the Banach-Alaoglu theorem, we see that the topology of pointwise convergence turns $\hat{A}$ into a compact Hausdorff space.
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**Definition**
For **commutative** \( A \), we define its **spectrum** (aka character space) as

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Due to the Banach-Alaoglu theorem, we see that the topology of pointwise convergence turns \( \hat{A} \) into a compact Hausdorff space.
Observation
If \( J \subset A \) is a maximal ideal in a (unital) Banach algebra, then \( J \) is closed. If \( A \) is commutative, then \( A/J \cong \mathbb{C} \) as a Banach algebra.

**Proof:** Part 1: Since the invertibles are open, there are no non-trivial dense ideals in \( A \). So \( \overline{J} \) is a proper ideal, hence \( J = \overline{J} \) by maximality.
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Part 2: The quotient is a Banach algebra in which every non-zero element is invertible. If it has a non-scalar element $x \in A/J$, then $\lambda - x \neq 0$ is invertible for all $\lambda \in \mathbb{C}$, which is a contradiction to $\sigma(x) \neq \emptyset$. 

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Observation
For commutative $A$, the assignment $\varphi \mapsto \ker \varphi$ is a 1-1 correspondence between $\hat{A}$ and maximal ideals in $A$.

Proof: Clearly the kernel of a character is a maximal ideal as it has codimension 1 in $A$. Since we have $\varphi(1) = 1$ for every $\varphi \in \hat{A}$ and $A = \mathbb{C}1 + \ker \varphi$, every character is uniquely determined by its kernel. Conversely, if $J \subset A$ is a maximal ideal, then $A/J \cong \mathbb{C}$, so the quotient map gives us a character.
\( A \) is still commutative.

**Theorem**

Let \( x \in A \). Then

\[ \sigma(x) = \left\{ \varphi(x) \mid \varphi \in \hat{A} \right\}. \]

**Proof:** Let \( \lambda \in \mathbb{C} \). If \( \lambda = \varphi(x) \), then \( \lambda - x \in \ker(\varphi) \), so \( \lambda - x \) is not invertible. Conversely, if \( \lambda - x \) is not invertible, then it is inside a (proper) maximal ideal. By the previous observation, this means \( (\lambda - x) \in \ker \varphi \) for some \( \varphi \in \hat{A} \), or \( \lambda = \varphi(x) \).
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**Theorem (Spectral radius formula)**

For any Banach algebra \( A \) and \( x \in A \), one has

\[
r(x) = \lim_{n \to \infty} \sqrt[n]{\|x^n\|}.
\]

**Proof:** The “\( \leq \)” part follows easily from the above (for \( A \) commutative). The “\( \geq \)” part is another clever application of complex analysis.
For commutative $A$, consider the usual embedding

$$\iota : A \hookrightarrow A^{**}, \quad \iota(x)(f) = f(x).$$

Since every element of $A^{**}$ is a continuous function on $\hat{A} \subset A^*$ in a natural way, we have a restriction mapping $A^{**} \rightarrow C(\hat{A})$. The composition of these two maps yields:
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**Definition (Gelfand transform)**

The Gelfand transform is the unital homomorphism $A \to C(\hat{A}), \ x \mapsto \hat{x}$ given by $\hat{x}(\varphi) = \varphi(x)$. 
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**Observation**

The Gelfand transform is norm-contractive. In fact, for $x \in A$ we have $\hat{x}(\hat{A}) = \sigma(x)$ and hence $\|\hat{x}\| = r(x) \leq \|x\|$ for all $x \in A$. 
Definition

Let $A$ be a unital $\mathbb{C}^*$-algebra. An element $x \in A$ is

1. **normal**, if $x^*x = xx^*$.
2. **self-adjoint**, if $x = x^*$.
3. **positive**, if $x = y^*y$ for some $y \in A$.
   Write $x \geq 0$.
4. **a unitary**, if $x^*x = xx^* = 1$. 


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\[ \text{positive } \rightarrow \text{ self-adjoint} \]
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Observation

Any element $x \in A$ can be written as $x = x_1 + ix_2$ for the self-adjoint elements

$$x_1 = \frac{x + x^*}{2}, \quad x_2 = \frac{x - x^*}{2i}.$$
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Observation

If $x \in A$ is self-adjoint, then it follows for all $t \in \mathbb{R}$ that

$$\|x + it\|^2 = \|(x - it)(x + it)\| = \|x^2 + t^2\| \leq \|x\|^2 + t^2.$$
**Proposition**

*If* \( x \in A \) *is self-adjoint, then* \( \sigma(x) \subseteq \mathbb{R} \).

**Proof:** Step 1: The spectrum of \( x \) inside \( A \) is the same as the spectrum of \( x \) inside its bicommutant \( A \cap \{ x \}'' \).\(^1\) As \( x \) is self-adjoint, this is a commutative \( C^* \)-algebra. So assume \( A \) is commutative.

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\(^1\)This holds in any Banach algebra.
Proposition

If $x \in A$ is self-adjoint, then $\sigma(x) \subset \mathbb{R}$.

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Step 2: For $\varphi \in \hat{A}$, we get

$$|\varphi(x) + it|^2 = |\varphi(x + it)^2| \leq \|x\|^2 + t^2, \quad t \in \mathbb{R}.$$ 

But this is only possible for $\varphi(x) \in \mathbb{R}$, as the left-hand expression will otherwise outgrow the right one as $t \to (\pm)\infty$.\(^2\)

---

\(^1\)This holds in any Banach algebra.

\(^2\)Notice: this works for any $\varphi \in A^*$ with $\|\varphi\| = \|\varphi(1)\| = 1$!
**Proposition**

Let $A$ be a commutative $C^*$-algebra. Then every character $\varphi \in \hat{A}$ is $\ast$-preserving, i.e., it satisfies $\varphi(x^*) = \overline{\varphi(x)}$ for all $x \in A$.

**Proof:** Write $x = x_1 + ix_2$ as before and use the above for

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\varphi(x^*) = \varphi(x_1 - ix_2) = \varphi(x_1) - i\varphi(x_2) = \overline{\varphi(x_1)} + i\overline{\varphi(x_2)} = \overline{\varphi(x)}.
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Corollary

For a commutative $\mathbb{C}^*$-algebra $A$, the Gelfand transform

$$A \to \mathbb{C}(\hat{A}), \quad \hat{x}(\varphi) = \varphi(x)$$

is a $\ast$-homomorphism.
Let $A$ be a $C^*$-algebra and $B \subseteq A$ a $C^*$-subalgebra.

**Observation**

An element $x \in A$ is invertible if and only if $x^*x$ and $xx^*$ are invertible.
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An element $x \in B$ is invertible in $B$ if and only if it is invertible in $A$.

**Proof:** By the above we may assume $x = x^*$. We know $\sigma_B(x) \subseteq \mathbb{R}$, so 
\[ x_n = x + \frac{i}{n} \quad n \to \infty \] is a sequence of invertibles in $B$. We know 
\[ \|x_n - x\| < \|x_n^{-1}\|^{-1} \] implies that $x$ is invertible in $B$. 

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**Corollary**

We have $\sigma_B(x) = \sigma_A(x)$ for all $x \in B$.

---

$^3$This often fails for inclusions of Banach algebras!
Let $A$ be a $C^*$-algebra.

**Observation**

$x \in A$ is normal if and only if $\mathcal{C}^*(x, 1) \subseteq A$ is commutative. In this case the spectrum of $\mathcal{C}^*(x, 1)$ is homeomorphic to $\sigma(x)$. 
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**Observation**

$x \in A$ is normal if and only if $C^*(x, 1) \subseteq A$ is commutative. In this case the spectrum of $C^*(x, 1)$ is homeomorphic to $\sigma(x)$.

**Proposition**

*For a normal element* $x \in A$, we have $r(x) = \|x\|$.

**Proof:** Observe from the $C^*$-identity that

$$\|x\|^4 = \|x^*x\|^2 = \|x^*xx^*x\| = \|(x^2)^*x^2\| = \|x^2\|^2.$$ 

By induction, we get $\|x^{2n}\| = \|x\|^{2n}$. By the spectral radius formula, we have

$$r(x) = \lim_{n \to \infty} 2^n \sqrt{\|x^{2n}\|} = \|x\|.$$
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\[ r(x) = \lim_{n \to \infty} \sqrt[n]{\|x^{2n}\|} = \|x\|. \]

**Corollary**

*For all $x \in A$, we have $\|x\| = \sqrt{\|x^*x\|} = \sqrt{r(x^*x)}$.***

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Theorem (Gelfand–Naimark)

For a commutative C*-algebra $A$, the Gelfand transform

$$A \rightarrow \mathcal{C}(\hat{A}), \quad \hat{x}(\varphi) = \varphi(x)$$

is an isometric *-isomorphism.

Proof: We have already seen that it is a *-homomorphism.
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is an isometric $\ast$-isomorphism.

Proof: We have already seen that it is a $\ast$-homomorphism. As every element $x \in A$ is normal, we have $\|x\| = r(x) = \|\hat{x}\|$, hence the Gelfand transform is isometric.
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is an isometric \( * \)-isomorphism.

Proof: We have already seen that it is a \( * \)-homomorphism. As every element \( x \in A \) is normal, we have \( \|x\| = r(x) = \|\hat{x}\| \), hence the Gelfand transform is isometric.

For surjectivity, observe that the image of \( A \) in \( C(\hat{A}) \) is a closed unital self-adjoint subalgebra, and which separates points. By the Stone–Weierstrass theorem, it follows that it is all of \( C(\hat{A}) \).
Observation

Let $x \in A$ be a normal element in a $C^*$-algebra. Let $A_x = C^*(x, 1)$ be the commutative $C^*$-subalgebra generated by $x$. Then $\hat{A}_x \cong \sigma(x)$ by observing that for every $\lambda \in \sigma(x)$ there is a unique $\varphi \in \hat{A}_x$ with $\varphi(x) = \lambda$. Under this identification $\hat{x} \in \mathcal{C}(\hat{A}_x)$ becomes the identity map on $\sigma(x)$. 
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**Theorem (functional calculus)**

Let $x \in A$ be a normal element in a (unital) C*-algebra. There exists a unique (isometric) $\ast$-homomorphism

\[ C(\sigma(x)) \to A, \quad f \mapsto f(x) \]

that sends $\text{id}_{\sigma(x)}$ to $x$.

**Proof:** Take the inverse of the Gelfand transform

\[ A_x \to C(\hat{A}_x) \cong C(\sigma(x)). \]
Theorem

An element \( x \in A \) is positive if and only if \( x \) is normal and \( \sigma(x) \subseteq \mathbb{R}^{\geq 0} \).

Proof: If the latter is true, then \( y = \sqrt{x} \) satisfies \( y^*y = y^2 = x \). So \( x \) is positive. The “only if” part is much trickier.
**Theorem**

An element $x \in A$ is positive if and only if $x$ is normal and $\sigma(x) \subseteq \mathbb{R}^{\geq 0}$.

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**Observation**

$x = x^* \in A$ is positive if and only if $\|r - x\| \leq r$ for some (or all) $r \geq \|x\|$.
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\( x = x^* \in A \) is positive if and only if \( \| r - x \| \leq r \) for some (or all) \( r \geq \| x \| \).

**Corollary**

For \( a, b \in A \) positive, the sum \( a + b \) is positive.

**Proof:** Apply the triangle inequality: We have \( \| a + b \| \leq \| a \| + \| b \| \) and

\[
\|(\| a \| + \| b \|) - (a + b)\| \leq \|\| a \| - a\| + \|\| b \| - b\| \leq \| a \| + \| b \|.
\]
Theorem

Every algebraic (unital) \(*\)-homomorphism \(\psi : A \to B\) between (unital) \(C^*\)-algebras is contractive, and hence continuous.\(^4\)

**Proof:** It is clear that \(\sigma(\psi(x)) \subseteq \sigma(x)\) for all \(x \in A\). By the spectral characterization of the norm, it follows that

\[
\|\psi(x)\|^2 = r(\psi(x^*x)) \leq r(x^*x) = \|x\|^2.
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\(^4\)This generalizes to the non-unital case as well!
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Observation

For \( x \in A \) normal and \( f \in \mathcal{C}(\sigma(x)) \), we have \( \psi(f(x)) = f(\psi(x)) \).

Proof: Clear for \( f \in \{\text{*polynomials}\} \). The general case follows by continuity of the assignments \([f \mapsto f(x)]\) and \([f \mapsto f(\psi(x))]\) and the Weierstrass approximation theorem.

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Theorem

Every injective $\ast$-homomorphism $\psi : A \to B$ is isometric.

Proof: By the $\mathbb{C}^*$-identity, it suffices to show $\|\psi(x)\| = \|x\|$ for positive $x \in A$. Suppose we have $\|\psi(x)\| < \|x\|$. Choose a non-zero continuous function $f : \sigma(x) \to \mathbb{R}_{\geq 0}$ with $f(\lambda) = 0$ for $\lambda \leq \|\psi(x)\|$.
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$$\psi(f(x)) = f(\psi(x)) = 0,$$

which means $\psi$ is not injective.
Definition

Let $A$ be a $C^*$-algebra. A **representation** (on a Hilbert space $\mathcal{H}$) is a $\ast$-homomorphism $\pi : A \to B(\mathcal{H})$. 
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1. **faithful**, if it is injective.

2. non-degenerate if $\text{span} \pi(A) \mathcal{H} = \mathcal{H}$.

3. cyclic, if there exists a vector $\xi \in \mathcal{H}$ with $\pi(A) \xi = \mathcal{H}$. For $\|\xi\| = 1$, we say that $(\pi, \mathcal{H}, \xi)$ is a cyclic representation.

4. irreducible, if $\pi(A) \xi = \mathcal{H}$ for all $0 \neq \xi \in \mathcal{H}$. 

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Gábor Szabó (KU Leuven)
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A functional $\varphi : A \to \mathbb{C}$ is called **positive**, if $\varphi(a) \geq 0$ whenever $a \geq 0$. 
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A functional $\varphi : A \to \mathbb{C}$ is called **positive**, if $\varphi(a) \geq 0$ whenever $a \geq 0$.

**Observation**

Every positive functional $\varphi : A \to \mathbb{C}$ is continuous.

**Proof:** Suppose not. By functional calculus, every element $x \in A$ can be written as a linear combination of at most four positive elements

$$x = (x_1^+ - x_1^-) + i(x_2^+ - x_2^-)$$

with norms $\|x_1^+\|, \|x_1^-\|, \|x_2^+\|, \|x_2^-\| \leq \|x\|$. So $\varphi$ is unbounded on the positive elements.
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Given $n \geq 1$, one may choose $a_n \geq 0$ with $\|a_n\| = 1$ and $\varphi(a_n) \geq n2^n$. Then $a = \sum_{n=1}^{\infty} 2^{-n}a_n$ is a positive element in $A$. By positivity of $\varphi$, we have $\varphi(a) \geq \varphi(2^{-n}a_n) \geq n$ for all $n$, a contradiction.
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Corollary

For a positive functional $\varphi$, the assignment $(x, y) \mapsto \varphi(y^* x)$ defines a positive semi-definite, anti-symmetric, sesqui-linear form. In particular, it is subject to the **Cauchy–Schwarz** inequality

$$|\varphi(y^* x)|^2 \leq \varphi(x^* x) \varphi(y^* y).$$
Theorem

Let $A$ be a unital $C^*$-algebra. A linear functional $\varphi : A \to \mathbb{C}$ is positive if and only if $\|\varphi\| = \varphi(1)$.

Proof: For the “only if” part, observe for $\|y\| \leq 1$ that

$$|\varphi(y)|^2 = |\varphi(1y)|^2 \leq \varphi(1)\varphi(y^*y) \leq \varphi(1)\|\varphi\|.$$

Taking the supremum over all such $y$ yields $\|\varphi\| = \varphi(1)$. 
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For the “if” part, suppose $\varphi(1) = 1 = \|\varphi\|$. Let $a \geq 0$ with $\|a\| \leq 1$. Repeating an argument we have used for characters, we know $\varphi(a) \in \mathbb{R}$. We have $\|1 - a\| \leq 1$. If $\varphi(a) < 0$, then it would necessarily follow that $\varphi(1 - a) = 1 - \varphi(a) > 1$, which contradicts $\|\varphi\| = 1$. Hence $\varphi(a) \geq 0$. Since $a$ was arbitrary, it follows that $\varphi$ is positive.
Corollary

For an inclusion of (unital) $C^*$-algebras $B \subseteq A$, every positive functional on $B$ extends to a positive functional on $A$.

Proof: Use Hahn–Banach and the previous slide.
Corollary

For an inclusion of (unital) $\mathbb{C}^*$-algebras $B \subseteq A$, every positive functional on $B$ extends to a positive functional on $A$.

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Definition

A state on a $\mathbb{C}^*$-algebra is a positive functional with norm one.
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Definition

A state on a $C^*$-algebra is a positive functional with norm one.

Observation

For $x \in A$ normal, there is a state $\varphi$ with $\|x\| = |\varphi(x)|$.

Proof: Pick $\lambda_0 \in \sigma(x)$ with $|\lambda_0| = \|x\|$. We know $A_x = C^*(x, 1) \cong C(\sigma(x))$ so that $x \mapsto \text{id}$. The evaluation map $f \mapsto f(\lambda_0)$ corresponds to a state on $A_x$ with the desired property. Extend it to a state $\varphi$ on $A$. 
Let $A$ be a $C^*$-algebra.

**Definition**

For self-adjoint elements $a, b \in A$, write $a \leq b$ if $b - a$ is positive.
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**Observation**

- The order “$\leq$” is compatible with sums.
- For all self-adjoint $a \in A$, we have $a \leq \|a\|$.
- If $a \leq b$ and $x \in A$ is any element, then $x^*ax \leq x^*bx$. 
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For proving the last part, write $b - a = c^*c$. Then

$$x^*bx - x^*ax = x^*(b - a)x = x^*c^*cx = (cx)^*cx \geq 0.$$
Given a state $\varphi$ on $A$, we have observed that $(x, y) \mapsto \varphi(y^*x)$ forms a positive semi-definite, anti-symmetric, sesqui-linear form.

**Observation**

For all $a, x \in A$, we have $\varphi(x^*a^*ax) \leq \|a\|^2 \varphi(x^*x)$. The null space $N_\varphi = \{x \in A \mid \varphi(x^*x) = 0\}$ is a closed left ideal in $A$. 
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**Observation**

The quotient $H_\varphi = A/N_\varphi$ carries the inner product

$$\langle [x] \mid [y] \rangle_\varphi = \varphi(y^*x),$$

and the left $A$-module structure satisfies $\|[ax]\|_\varphi \leq \|a\| \cdot \|[x]\|_\varphi$ for all $a, x \in A$. 
Definition (Gelfand–Naimark–Segal construction)

For a state $\varphi$ on a C*-algebra $A$, let $\mathcal{H}_\varphi$ be the Hilbert space completion $\mathcal{H}_\varphi = \overline{H_\varphi \| \cdot \|_\varphi}$. Then $\mathcal{H}_\varphi$ carries a unique left $A$-module structure which extends the one on $H_\varphi$ and is continuous in $\mathcal{H}_\varphi$. This gives us a representation

$$\pi_\varphi : A \to B(\mathcal{H}_\varphi) \text{ via } \pi_\varphi(a)([x]) = [ax]$$

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The only non-tautological part is that $\pi_\varphi$ is compatible with adjoints. For this we observe

$$\langle [ax] | [y] \rangle_\varphi = \varphi(y^* ax) = \varphi((a^* y)^* x) = \langle [x] | [a^* y] \rangle_\varphi,$$

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Definition

In the (unital) situation above, set $\xi_\varphi = [1] \in H_\varphi$. Then $\|\xi_\varphi\| = 1$ as we have assumed $\varphi$ to be a state.
**Theorem (GNS)**

The assignment \( \varphi \mapsto (\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi) \) is a 1-1 correspondence between states on \( A \) and cyclic representations modulo unitary equivalence.

**Proof:** Let us only check that \( (\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi) \) is cyclic. Indeed,

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\pi_\varphi(A)\xi_\varphi = \pi_\varphi(A)([1]) = [A] = H_\varphi \subseteq \mathcal{H}_\varphi,
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Theorem (Gelfand–Naimark)

Every abstract \( \mathbb{C}^* \)-algebra \( A \) is a concrete \( \mathbb{C}^* \)-algebra. In particular, there exists a faithful representation \( \pi : A \to \mathcal{H} \) on some Hilbert space.

Proof: For \( x \in A \), find \( \varphi_x \) with \( \| \varphi_x(x^*x) \| = \| x \|^2 \). Then form the cyclic representation \( (\pi_{\varphi_x}, \mathcal{H}_{\varphi_x}, \xi_{\varphi_x}) \).

\[5\] If \( A \) is separable, we may choose \( \mathcal{H} \) to be separable!
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Theorem (Gelfand–Naimark)

Every abstract $C^*$-algebra $A$ is a concrete $C^*$-algebra. In particular, there exists a faithful representation $\pi: A \to \mathcal{H}$ on some Hilbert space.$^5$

Proof: For $x \in A$, find $\varphi_x$ with $\|\varphi_x(x^*x)\| = \|x\|^2$. Then form the cyclic representation $(\pi_{\varphi_x}, \mathcal{H}_{\varphi_x}, \xi_{\varphi_x})$. We claim that the direct sum

$$\pi := \bigoplus_{x \in A} \pi_{\varphi_x} : A \to B\left( \bigoplus_{x \in A} \mathcal{H}_{\varphi_x} \right)$$

does it. Indeed, given any $x \neq 0$ we have

$$\|\pi(x)\|^2 \geq \|\pi(x)\xi_{\varphi_x}\|^2 = \langle [x] | [x] \rangle_{\varphi_x} = \varphi_x(x^*x) = \|x\|^2.$$

$^5$If $A$ is separable, we may choose $\mathcal{H}$ to be separable!
Let us now discuss noncommutative examples of $\mathbb{C}^*$-algebras:

**Example**

The set of $\mathbb{C}$-valued $n \times n$ matrices, denoted $M_n$, becomes a $\mathbb{C}^*$-algebra. By linear algebra, $M_n \cong \mathcal{B}(\mathbb{C}^n)$. 

Let us now discuss noncommutative examples of C*-algebras:

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**Example**

For numbers \( n_1, \ldots, n_k \geq 1 \), the C*-algebra

\[ A = M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_k} \]

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**Theorem**

*Every finite-dimensional \( \mathbb{C}^* \)-algebras has this form.*
Recall

A linear map between Banach spaces $T : A \to B$ is called compact, if $T \cdot A_{\|\cdot\| \leq 1} \subseteq B$ is compact.
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For a Hilbert space $\mathcal{H}$, the set of compact operators $\mathcal{K}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$ forms a norm-closed, $*$-closed, two-sided ideal. If $\dim(\mathcal{H}) = \infty$, then it is a proper ideal and a non-unital $C^*$-algebra.
Notation (ad-hoc!)

Let $\mathcal{G}$ be a countable set, and let $\mathcal{P}$ be a family of (noncommutative) $\ast$-polynomials in finitely many variables in $\mathcal{G}$ and coefficients in $\mathbb{C}$. We shall understand a relation $\mathcal{R}$ as a collection of formulas of the form

$$\|p(\mathcal{G})\| \leq \lambda_p, \quad p \in \mathcal{P}, \quad \lambda_p \geq 0.$$ 

A representation of $(\mathcal{G} \mid \mathcal{R})$ is a map $\pi : \mathcal{G} \to A$ into a $\mathbb{C}^*$-algebra under which the relation becomes true.
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Example

The expression $xyx^* - z^2$ for $x, y, z \in \mathcal{G}$ is a noncommutative $\ast$-polynomial. The relation could mean

$$\|xyx^* - z^2\| \leq 1.$$
A representation $\pi_u$ of $(\mathcal{G} | \mathcal{R})$ into a $C^*$-algebra $B$ is called universal, if

$$B = C^*(\pi_u(\mathcal{G})).$$

whenever $\pi: G \to A$ is a representation of $(\mathcal{G} | \mathcal{R})$ into another $C^*$-algebra, there exists a $\ast$-homomorphism $\phi: B \to A$ such that $\phi \circ \pi_u = \pi$. Up to isomorphism, a $C^*$-algebra $B$ as above is unique. One writes $B = C^*(\mathcal{G})$ and calls it the universal $C^*$-algebra for $(\mathcal{G} | \mathcal{R})$. 

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Up to isomorphism, a $C^*$-algebra $B$ as above is unique. One writes $B = C^*(\mathcal{G} \mid \mathcal{R})$ and calls it the universal $C^*$-algebra for $(\mathcal{G} \mid \mathcal{R})$. 
Example

Given \( n \geq 1 \), one can express \( M_n \) as the universal \( \mathbb{C}^* \)-algebra generated by \( \{ e_{i,j} \}_{i,j=1}^n \) subject to the relations

\[
e_{ij} e_{kl} = \delta_{jk} e_{il}, \quad e_{ij}^* = e_{ji}.
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Example

Given $n \geq 1$, one can express $M_n$ as the universal $\mathbb{C}^*$-algebra generated by $\{e_{i,j}\}_{i,j=1}^n$ subject to the relations

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Example

Let $\mathcal{H}$ be a separable, infinite-dimensional Hilbert space. Then one can express $\mathcal{K}(\mathcal{H})$ as the universal $\mathbb{C}^*$-algebra generated by $\{e_{i,j}\}_{i,j \in \mathbb{N}}$ subject to the relations

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$$e_{ij}e_{kl} = \delta_{jk}e_{il}, \quad e_{ij}^* = e_{ji}.$$ 

(Here $e_{ij}$ represents a rank-one operator sending the $i$-th vector in an ONB to the $j$-th vector.)
**Definition**

A relation $\mathcal{R}$ on a set $\mathcal{G}$ is **compact** if for every $x \in \mathcal{G}$

$$\sup \{ \| \pi(x) \| \mid \pi : \mathcal{G} \to A \text{ representation of } (\mathcal{G} \mid \mathcal{R}) \} < \infty.$$
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Theorem

*For a pair $(\mathcal{G} \mid \mathcal{R})$, the universal $C^*$-algebra $C^*(\mathcal{G} \mid \mathcal{R})$ exists if and only if $\mathcal{R}$ is compact.*

**Proof:** The “only if” part follows from the fact that $*$-homomorphisms are contractive.
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For a pair $(\mathcal{G} | \mathcal{R})$, the universal $C^*$-algebra $C^*(\mathcal{G} | \mathcal{R})$ exists if and only if $\mathcal{R}$ is compact.

**Proof:** The “only if” part follows from the fact that $*$-homomorphisms are contractive.

“if” part: The isomorphism classes of separable $C^*$-algebras form a set. There exist set-many representations $\pi : \mathcal{G} \to A_\pi$ of $(\mathcal{G} | \mathcal{R})$ on separable $C^*$-algebras up to conjugacy. Denote this set by $I$, and consider

$$\mathcal{A} = \prod_{\pi \in I} A_\pi \quad \text{and} \quad \pi_u : \mathcal{G} \to \mathcal{A}, \; \pi_u(x) = (\pi(x))_{\pi \in I}.$$ 

By compactness, $\pi_u$ is a well-defined representation of $(\mathcal{G} | \mathcal{R})$. Then check that $B = C^*(\pi_u(\mathcal{G})) \subseteq \mathcal{A}$ is universal.
Example

The universal C*-algebra for the relation $\|xyx^* - z^2\| \leq 1$ does not exist.

**Proof:** Suppose we have such $x, y, z \neq 0$ in a C*-algebra, e.g., all equal to the unit. For $\lambda > 0$, replace $y \rightarrow \lambda y$ and $x \rightarrow \lambda^{-1/2}x$, and let $\lambda \rightarrow \infty$. 
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**Remark (Warning!)**

It can easily happen that a relation is compact and non-trivial, but the universal C*-algebra is zero! E.g., \( C^*(x \mid x^*x = -xx^*) = 0 \).
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Example

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C^*(u \mid u^*u = uu^* = 1) \cong C(\mathbb{T}) \quad \text{with} \quad u \mapsto \text{id}_\mathbb{T}.
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**Proof**: Functional calculus.
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**Example**

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**Proof:** Functional calculus.

**Remark**

All of this generalizes to more general relations (including functional calculus etc.) and a more flexible notion of generating sets.
**Proposition**

*Every separable C*-algebra* $A$ *is the universal C*-algebra for a countable set of equations involving*-polynomials of degree at most 2.*
**Proposition**

*Every separable C*-algebra A is the universal C*-algebra for a countable set of equations involving *-polynomials of degree at most 2.*

**Proof:** Start with some countable dense \( \mathbb{Q}[i]-\)*-subalgebra \( C \subset A \). By inductively enlarging \( C \), we may enlarge it to another countable dense \( \mathbb{Q}[i]-\)*-subalgebra \( D \subset A \) with the additional property that if \( x \in D \) is a contraction, then \( y = 1 - \sqrt{1 - x^*x} \in D \).
**Proposition**

Every separable $\mathbb{C}^*$-algebra $A$ is the universal $\mathbb{C}^*$-algebra for a countable set of equations involving $\ast$-polynomials of degree at most 2.

**Proof:** Start with some countable dense $\mathbb{Q}[i]$-$\ast$-subalgebra $C \subset A$. By inductively enlarging $C$, we may enlarge it to another countable dense $\mathbb{Q}[i]$-$\ast$-subalgebra $D \subset A$ with the additional property that if $x \in D$ is a contraction, then $y = 1 - \sqrt{1 - x^*x} \in D$.

Now let $\mathcal{P}$ be the family of $\ast$-polynomials that encode all the $\ast$-algebra relations in $D$, so

$$X_aX_b - X_{ab}, \lambda X_a + X_b - X_{\lambda a+b}, X_a^* - X_{a^*},$$

for $\lambda \in \mathbb{Q}[i]$ and $a, b \in D$. Set $\mathcal{G} = D$, and let $\mathcal{R}$ be the relation where these polynomials evaluate to zero. By construction, representations $(\mathcal{G} | \mathcal{R}) \rightarrow B$ are the same as $\ast$-homomorphisms $D \rightarrow B$. 

Gábor Szabó (KU Leuven)
Proposition

Every separable \( C^\ast \)-algebra \( A \) is the universal \( C^\ast \)-algebra for a countable set of equations involving \( \ast \)-polynomials of degree at most 2.

Proof: (continued) By construction, representations \((\mathcal{G} \mid \mathcal{R}) \to B\) are the same as \( \ast \)-homomorphisms \( D \to B \).

We claim that the inclusion \( D \subset A \) turns \( A \) into the universal \( C^\ast \)-algebra for these relations. This means that every \( \ast \)-homomorphism from \( D \) extends to a \( \ast \)-homomorphism on \( A \). This is certainly the case if every \( \ast \)-homomorphism \( \varphi : D \to B \) is contractive.
**Proposition**

*Every separable $C^*$-algebra $A$ is the universal $C^*$-algebra for a countable set of equations involving $*$-polynomials of degree at most 2.*

**Proof: (continued)** By construction, representations $(G | R) \rightarrow B$ are the same as $*$-homomorphisms $D \rightarrow B$.

We claim that the inclusion $D \subset A$ turns $A$ into the universal $C^*$-algebra for these relations. This means that every $*$-homomorphism from $D$ extends to a $*$-homomorphism on $A$. This is certainly the case if every $*$-homomorphism $\varphi : D \rightarrow B$ is contractive.

Indeed, if $x \in D$ is a contraction, then $y = 1 - \sqrt{1 - x^*x} \in D_{sa}$ satisfies

$$x^*x + y^2 - 2y = 0.$$ 

Thus also $\varphi(x)^*\varphi(x) + \varphi(y)^2 - 2\varphi(y) = 0$ in $B$, which is equivalent to

$$\varphi(x)^*\varphi(x) + (1 - \varphi(y))^2 = 1.$$ 

Hence $\|\varphi(x)\| \leq 1$ for every contraction $x \in D$, which finishes the proof.
Definition

Let $\Gamma$ be a countable discrete group. The universal group $C^*$-algebra is defined as

$$C^*(\Gamma) = C^*(\{u_g\}_{g \in \Gamma} \mid u_1 = 1, u_{gh} = u_g u_h, u_g^* = u_{g^{-1}}).$$
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**Example**

$$C^*(\mathbb{Z}) \cong C(\mathbb{T}).$$
Definition

Let $\Gamma$ be a countable discrete group. The universal group $C^*$-algebra is defined as

$$C^* (\Gamma) = C^* \left( \{ u_g \}_{g \in \Gamma} \mid u_1 = 1, \ u_{gh} = u_g u_h, \ u_g^* = u_{g^{-1}} \right).$$

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Example

The Toeplitz algebra is $\mathcal{T} = C^* \left( s \mid s^* s = 1 \right).$
Definition

Let $\Gamma$ be a countable discrete group. The **universal group $C^*$-algebra** is defined as

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Example

The **Toeplitz algebra** is $\mathcal{T} = C^*(s \mid s^* s = 1)$.

Fact

*If $\nu \in B$ is any non-unitary isometry in a $C^*$-algebra, then $C^*(\nu) \cong \mathcal{T}$ in the obvious way. In other words, every proper isometry is universal.*
Example

For \( n \in \mathbb{N} \), one defines the **Cuntz algebra** in \( n \) generators as

\[
\mathcal{O}_n = \mathbb{C}^* \left( s_1, \ldots, s_n \mid s_j^* s_j = 1, \sum_{j=1}^{n} s_j s_j^* = 1 \right).
\]
Example

For $n \in \mathbb{N}$, one defines the **Cuntz algebra** in $n$ generators as

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\mathcal{O}_n = C^* \left( s_1, \ldots, s_n \mid s_j^* s_j = 1, \sum_{j=1}^{n} s_j s_j^* = 1 \right).
$$

\[\mathcal{O}_3 = C^*(s_1, s_2, s_3)\]

$$
\mathcal{H}_j = s_j \mathcal{H} \subseteq \mathcal{H}
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Example

For $n \in \mathbb{N}$, one defines the Cuntz algebra in $n$ generators as

$$\mathcal{O}_n = \mathbb{C}^\ast \left(s_1, \ldots, s_n \mid s_j^* s_j = 1, \sum_{j=1}^{n} s_j s_j^* = 1\right).$$

$\mathcal{O}_3 = \mathbb{C}^\ast(s_1, s_2, s_3)$

$H_j = s_j H \subseteq H$

Theorem (Cuntz)

$\mathcal{O}_n$ is simple! That is, every collection of isometries $s_1, \ldots, s_n$ in any $\mathbb{C}^\ast$-algebra as above is universal with this property.
Fact (Inductive limits)

If

\[ A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \]

is a sequence of C*-algebra inclusions, then

\[ A = \bigcup_{n \in \mathbb{N}} A_n \| \cdot \| \]

exists and is a C*-algebra.

Definition

In the above situation, if every \( A_n \) is finite-dimensional, we call \( A \) an AF algebra. (AF = approximately finite-dimensional)
Example

Consider

\[ A_1 = \mathbb{C}, \quad A_2 = M_2, \quad A_3 = M_4 \cong M_2 \otimes M_2, \quad A_4 = M_8 \cong M_2 \otimes M_2, \quad \ldots, \]

with inclusions of the form \( x \mapsto x \otimes 1_2 = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \).
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The \textbf{CAR algebra} is the limit

\[ M_{2^\infty} = M_2^{\otimes \infty} = \bigcup A_n. \]
Example

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The **CAR algebra** is the limit

\[ M_{2\infty} = M_2 \otimes^\infty = \bigcup A_n. \]

This construction can of course be repeated with powers of any other number \( p \) instead of 2. \( \leadsto M_{p\infty} \)
$M_2$
$M_4$
$M_8$
$M_{2\infty}$
Let $A$ be a (unital) $C^*$-algebra and $\Gamma$ a discrete group.

**Definition**

Given an action $\alpha : \Gamma \curvearrowright A$, define the **crossed product** $A \rtimes_\alpha \Gamma$ as the universal $C^*$-algebra containing a unital copy of $A$, and the image of a unitary representation $[g \mapsto u_g]$ of $\Gamma$, subject to the relation

$$u_g a u_g^* = \alpha_g(a), \quad a \in A, \ g \in \Gamma.$$
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**Example**

Start from a homeomorphic action $\Gamma \curvearrowright X$ on a compact Hausdorff space. $\sim \mathcal{C}(X) \rtimes \Gamma$. 

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Gábor Szabó (KU Leuven)  

C*-algebras  

November 2018
Observation

For two \( \mathcal{C}^* \)-algebras \( A, B \), the algebraic tensor product \( A \odot B \) becomes a \( * \)-algebra in the obvious way.

Question

Can this be turned into a \( \mathcal{C}^* \)-algebra?
Observation

For two $\mathcal{C}^*$-algebras $A, B$, the algebraic tensor product $A \odot B$ becomes a $\ast$-algebra in the obvious way.

Question

Can this be turned into a $\mathcal{C}^*$-algebra?

Yes! However, not uniquely in general.
Observation

For two $C^*$-algebras $A, B$, the algebraic tensor product $A \otimes B$ becomes a $*$-algebra in the obvious way.

Question

Can this be turned into a $C^*$-algebra?

Yes! However, not uniquely in general.

Definition

We say that a $C^*$-algebra $A$ is **nuclear** if the tensor product $A \otimes B$ carries a unique $C^*$-norm for every $C^*$-algebra $B$. In this case we denote by $A \otimes B$ the $C^*$-algebra arising as the completion.
Example

Finite-dimensional or commutative $C^*$-algebras are nuclear. One has $M_n \otimes A \cong M_n(A)$ and $C(X) \otimes A \cong C(X, A)$. 
Example

Finite-dimensional or commutative C*-algebras are nuclear. One has $M_n \otimes A \cong M_n(A)$ and $C(X) \otimes A \cong C(X, A)$.

Theorem

A discrete group $\Gamma$ is **amenable** if and only if $C^*(\Gamma)$ is nuclear.
Example

Finite-dimensional or commutative $\mathbb{C}^*$-algebras are nuclear. One has $M_n \otimes A \cong M_n(A)$ and $\mathcal{C}(X) \otimes A \cong \mathcal{C}(X, A)$.

Theorem

A discrete group $\Gamma$ is **amenable** if and only if $\mathbb{C}^*(\Gamma)$ is nuclear.

Example (free groups)

$\mathbb{C}^*(F_n)$ is not nuclear for $n \geq 2$. 
Example

Finite-dimensional or commutative $\mathcal{C}^*$-algebras are nuclear. One has $M_n \otimes A \cong M_n(A)$ and $\mathcal{C}(X) \otimes A \cong \mathcal{C}(X, A)$.

Theorem

A discrete group $\Gamma$ is **amenable** if and only if $\mathcal{C}^*(\Gamma)$ is nuclear.

Example (free groups)

$\mathcal{C}^*(F_n)$ is not nuclear for $n \geq 2$.

Theorem

*If* $\Gamma$ *is amenable and* $A$ *is nuclear, then* $A \rtimes \Gamma$ *is nuclear for every possible action* $\Gamma \curvearrowright A$. *So in particular for* $A = \mathcal{C}(X)$. 
Fact (K-theory)

There is a functor

\[ \{ \text{C}^*\text{-algebras} \} \longrightarrow \{ \text{abelian groups} \}, \quad A \mapsto K_* (A) = K_0 (A) \oplus K_1 (A), \]

which extends the topological $K$-theory functor $X \mapsto K^* (X)$ for (locally) compact Hausdorff spaces.
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\( K_0(A) \) has a natural \textbf{positive part} \( K_0(A)_+ \), which induces an order relation on \( K_0(A) \).
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There is a functor

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Fact

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Theorem (Glimm, Bratteli, Elliott)

Let \(A\) and \(B\) be two (unital) AF algebras. Then

\[
A \cong B \iff (K_0(A), K_0(A)_+, [1_A]) \cong (K_0(B), K_0(B)_+, [1_B]).
\]
Definition (Elliott invariant)

For a (unital) simple $C^*$-algebra $A$, one considers

- its $K$-groups $K_0(A)$ and $K_1(A)$;
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- a natural pairing map $\rho_A : T(A) \times K_0(A) \to \mathbb{R}$ which is an order homomorphism in the second variable.
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The sextuple

$$\text{Ell}(A) = \left( K_0(A), K_0(A)_+, [1_A], K_1(A), T(A), \rho_A \right)$$

is called the \textbf{Elliott invariant} and becomes functorial with respect to a suitable target category.
Fact

There is a separable unital simple nuclear infinite-dimensional $\mathbb{C}^*$-algebra $\mathcal{Z}$ with $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z}$, the Jiang–Su algebra, with $\text{Ell}(\mathcal{Z}) \cong \text{Ell}(\mathbb{C})$. 

Rough idea: One considers the $\mathbb{C}^*$-algebra $\mathcal{Z}_2^\infty$, $\mathcal{Z}_3^\infty = \{ f \in C([0,1], M_2^\infty \otimes M_3^\infty \mid f(0) \in M_2^\infty \otimes 1, f(1) \in 1 \otimes M_3^\infty \}$, which has the right $K$-theory but far too many ideals and traces. One constructs a trace-collapsing endomorphism on $\mathcal{Z}_2^\infty$, $\mathcal{Z}_3^\infty$ and can define $\mathcal{Z}$ as the stationary inductive limit.
Fact

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Rough idea: One considers the $C^*$-algebra

$$\mathcal{Z}_{2\infty,3\infty} = \{ f \in C([0,1], M_{2\infty} \otimes M_{3\infty}) \mid f(0) \in M_{2\infty} \otimes 1, \ f(1) \in 1 \otimes M_{3\infty} \}$$

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which has the right $K$-theory but far too many ideals and traces.

One constructs a **trace-collapsing** endomorphism on $\mathcal{Z}_{2\infty,3\infty}$ and can define $\mathcal{Z}$ as the stationary inductive limit.

(Graphic created by Aaron Tikuisis.)
Definition

We say that a $C^*$-algebra $A$ is $\mathcal{Z}$-stable, if $A \cong A \otimes \mathcal{Z}$. 
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*If $A$ is simple and the order on $K_0(A)$ satisfies a mild condition, then $\text{Ell}(A) \cong \text{Ell}(A \otimes \mathcal{Z})$.***
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Fact
*If* $A$ *is simple and the order on* $K_0(A)$ *satisfies a mild condition, then*
$\text{Ell}(A) \cong \text{Ell}(A \otimes \mathcal{Z})$.

Conjecture (Elliott conjecture; modern version)
Let $A$ and $B$ be two separable unital simple nuclear $\mathcal{Z}$-stable $C^*$-algebras. Then

$$A \cong B \iff \text{Ell}(A) \cong \text{Ell}(B).$$

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6To the experts in the audience: No UCT discussion now!
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If $A$ is simple and the order on $K_0(A)$ satisfies a mild condition, then $\text{Ell}(A) \cong \text{Ell}(A \otimes \mathcal{Z})$.

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Let $A$ and $B$ be two separable unital simple nuclear $\mathcal{Z}$-stable $C^*$-algebras. Then

$$A \cong B \iff \text{Ell}(A) \cong \text{Ell}(B).$$

(There is a more general version not assuming unitality.)

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$^6$To the experts in the audience: No UCT discussion now!
Definition

We say that a C*-algebra $A$ is $\mathcal{Z}$-stable, if $A \cong A \otimes \mathcal{Z}$.

Fact

If $A$ is simple and the order on $K_0(A)$ satisfies a mild condition, then $\text{Ell}(A) \cong \text{Ell}(A \otimes \mathcal{Z})$.

Conjecture (Elliott conjecture; modern version)

Let $A$ and $B$ be two separable unital simple nuclear $\mathcal{Z}$-stable C*-algebras. Then

$$A \cong B \iff \text{Ell}(A) \cong \text{Ell}(B).$$

(There is a more general version not assuming unitality.)

Problem (difficult!)

Determine when $\Gamma \rtimes X$ gives rise to a $\mathcal{Z}$-stable crossed product.

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Thank you for your attention!