The stable uniqueness theorem for equivariant Kasparov theory

Workshop C*-algebras, Oberwolfach

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KU Leuven

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Question

What should this even mean when we classify up to cocycle conjugacy?

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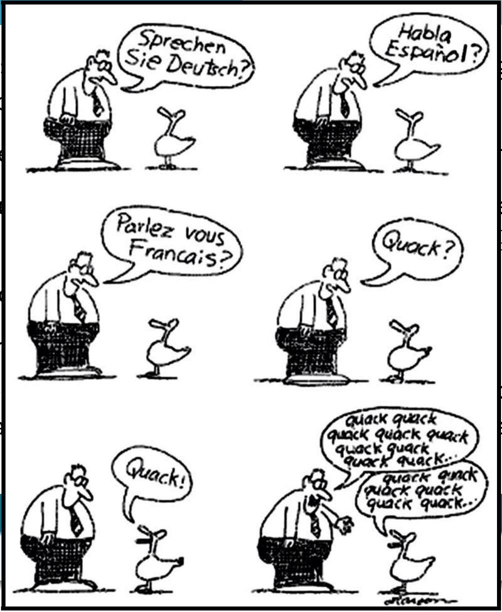

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$\rightsquigarrow$ We need the appropriate language

Definition

Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be two actions on C^*-algebras. A cocycle representation is a pair

$$(\varphi, \mathbb{u}) : (A, \alpha) \to (\mathcal{M}(B), \beta),$$

where $\varphi : A \to \mathcal{M}(B)$ is a $*$-homomorphism, $\mathbb{u} : G \to \mathcal{U}(\mathcal{M}(B))$ is a β-cocycle, and we have $\mathrm{Ad}(\mathbb{u}_g) \circ \beta_g \circ \varphi = \varphi \circ \alpha_g$ for all $g \in G$.
If $\varphi(A) \subseteq B$, then $(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta)$ is called a cocycle morphism.

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Example

For $\mathbb{u} = \mathbf{1}$, we recover what it means for φ to be equivariant.
For $\beta = \mathrm{id}$, we recover the concept of a covariant representation for (A, α).

Definition (Composition)

Let $\alpha : G \curvearrowright A$, $\beta : G \curvearrowright B$, and $\gamma : G \curvearrowright C$ be three actions on C^*-algebras. Suppose that

$$(\varphi, \mathbb{u}) : (A, \alpha) \to (\mathcal{M}(B), \beta) \quad \text{and} \quad (\psi, \mathbb{v}) : (B, \beta) \to (\mathcal{M}(C), \gamma)$$

are two (non-degenerate) cocycle representations. Then the pair

$$(\psi \circ \varphi, \psi(\mathbb{u}_\bullet)\mathbb{v}_\bullet) =: (\psi, \mathbb{v}) \circ (\varphi, \mathbb{u})$$

is a cocycle representation from (A, α) to $(\mathcal{M}(C), \gamma)$.

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The binary operation "$\circ$" becomes associative, and on every object (A, α) the pair $(\mathrm{id}_A, \mathbf{1}) = \mathrm{id}_A$ is a neutral element. Thus we can consider the G-C^*-algebras as a category with morphisms being the cocycle morphisms.

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Observation

An isomorphism in this category is precisely a cocycle conjugacy.

Example

For a given action $\beta : G \curvearrowright B$ and a unitary $u \in \mathcal{U}(\mathcal{M}(B))$, the pair

$$\mathrm{Ad}(u) := (\mathrm{Ad}(u), u\beta_\bullet(u)^*)$$

is an inner cocycle morphism on (B, β).

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Remark

We can equip the set of cocycle morphisms $(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta)$ with the point-norm topology in the first variable, and the strict topology in the second variable, but uniformly over compact sets $K \subseteq G$. If A is separable and B is σ-unital, then this yields a Polish topology.

Definition

We say that a cocycle morphism $(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta)$ is approximately unitarily equivalent to $(\psi, \mathbb{v})$, if there exists a net $u_\lambda \in \mathcal{U}(\mathcal{M}(B))$ such that $\mathrm{Ad}(u_\lambda) \circ (\varphi, \mathbb{u}) \xrightarrow{\lambda \to \infty} (\psi, \mathbb{v})$. We write $(\varphi, \mathbb{u}) \approx_{\mathrm{u}} (\psi, \mathbb{v})$.

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Theorem (S; Elliott in unital case)

Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions on separable C^*-algebras. Suppose that

$$(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta) \quad \textit{and} \quad (\psi, \mathbb{v}) : (B, \beta) \to (A, \alpha)$$

are two cocycle morphisms such that

$$(\psi, \mathbb{v}) \circ (\varphi, \mathbb{u}) \approx_{\mathrm{u}} \mathrm{id}_A \quad \textit{and} \quad (\varphi, \mathbb{u}) \circ (\psi, \mathbb{v}) \approx_{\mathrm{u}} \mathrm{id}_B \, .$$

Then $(\varphi, \mathbb{u})$ and $(\psi, \mathbb{v})$ are approximately unitarily equivalent to mutually inverse cocycle conjugacies between (A, α) and (B, β).

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More specifically, we focus on Kasparov's G-equivariant KK-functor as an important invariant, and investigate what information we can extract.

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To every pair of actions $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ on separable C^*-algebras, we can assign an abelian group $KK^G(\alpha, \beta)$. This assignment is contravariant in the first variable and covariant in the second.

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The Kasparov product allows one to view the G-C^*-algebras as a new category, with $KK^G(\alpha, \beta)$ being the Hom-set of arrows $\alpha \to \beta$.

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Theorem (Thomsen, generalizing Cuntz and Higson)

The KK^G-category is universal for functors from separable G-C^*-algebras to abelian groups that are stable, half-exact, and homotopy invariant.

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The KK^G-category is universal for functors from separable G-C^*-algebras to abelian groups that are stable, half-exact, and homotopy invariant.

The key towards the proof of this is a generalization of the Cuntz picture of ordinary KK-theory. (*Cuntz–Thomsen picture*)

From now on, we will fix actions $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ on separable C^*-algebras, and assume (B, β) is conjugate to $(B \otimes \mathcal{K}, \beta \otimes \mathrm{id}_{\mathcal{K}})$.

Definition (Thomsen)

An (α, β)-Cuntz pair is a pair of cocycle representations

$$(\varphi, \mathbb{u}), (\psi, \mathbb{v}) : (A, \alpha) \to (\mathcal{M}(B), \beta)$$

such that the pointwise differences $\psi - \varphi$ and $\mathbb{v} - \mathbb{u}$ take values in B. We say that this pair is degenerate if $\varphi = \psi$.

(In the original definition, $\mathbb{v} - \mathbb{u}$ is assumed to be norm-continuous, which turns out to be automatic.)

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Definition

Pick two isometries $s_1, s_2 \in \mathcal{M}(\mathcal{K}) \subseteq \mathcal{M}(B)^\beta$ with $s_1 s_1^* + s_2 s_2^* = \mathbf{1}$. For $b_1, b_2 \in \mathcal{M}(B)$, one defines $b_1 \oplus b_1 = b_1 \oplus_{s_1,s_2} b_2 = s_1 b_1 s_1^* + s_2 b_2 s_2^*$. This element does not depend on the choice of s_1, s_2 up to conjugation with a uniquely determined unitary in $\mathcal{U}_0(\mathcal{M}(B)^\beta)$.

Definition

Given two (α, β)-Cuntz pairs $[(\varphi^{(j)}, \mathbb{u}^{(j)}), (\psi^{(j)}, \mathbb{v}^{(j)})]$ for $j = 1, 2$, we can define their sum as

$$\begin{aligned} & [(\varphi^{(1)}, \mathbb{u}^{(1)}), (\psi^{(1)}, \mathbb{v}^{(1)})] \oplus [(\varphi^{(2)}, \mathbb{u}^{(2)}), (\psi^{(2)}, \mathbb{v}^{(2)})] \\ = \quad & [(\varphi^{(1)} \oplus \varphi^{(2)}, \mathbb{u}^{(1)} \oplus \mathbb{u}^{(2)}), (\psi^{(1)} \oplus \psi^{(2)}, \mathbb{v}^{(1)} \oplus \mathbb{v}^{(2)})]. \end{aligned}$$

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Definition

For a $(\alpha,\beta[0,1])$-Cuntz pair

$$(\Phi,\mathbb{U}),(\Psi,\mathbb{V}):(A,\alpha)\to(\mathcal{M}(B[0,1]),\beta[0,1]),$$

the evaluation at the endpoints $0,1\in[0,1]$ yields two (α,β)-Cuntz pairs. This defines the homotopy relation $\sim_h$ on (α,β)-Cuntz pairs.

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The (α, β)-Cuntz pairs modulo homotopy form an abelian semigroup.

We denote $\mathbb{E}^G(\alpha,\beta) = \{(\alpha,\beta)\text{-Cuntz pairs}\}$, and $\mathbb{D}^G(\alpha,\beta)$ the subset given by degenerate elements. One defines an equivalence relation on $\mathbb{E}^G(\alpha,\beta)$ via

$$x_1 \sim_{sh} x_2 \quad :\Leftrightarrow \quad \exists\ d_1, d_2 \in \mathbb{D}^G(\alpha,\beta) :\ x_1 \oplus d_1 \sim_h x_2 \oplus d_2.$$

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Question (Stable uniqueness)

If a Cuntz pair

$$(\varphi,\mathbb{u}),(\psi,\mathbb{v}) : (A,\alpha) \to (\mathcal{M}(B),\beta)$$

defines the zero element in KK^G, what does this really tell us?

Definition

For two cocycle representations

$$(\varphi, \mathbb{u}), (\psi, \mathbb{v}) : (A, \alpha) \to (\mathcal{M}(B), \beta),$$

let us write $(\varphi, \mathbb{u}) \sim_B (\psi, \mathbb{v})$, if there is a continuous family $\{v_t\}_{t\in\mathbb{R}}$ in $\mathcal{U}(\mathcal{M}(B))$ such that $\mathrm{Ad}(v_t) \circ \varphi \xrightarrow{t\to\infty} \psi$ in point-norm, $v_t \mathbb{u}_g \beta_g(v_t)^* \xrightarrow{t\to\infty} \mathbb{v}_g$ in norm uniformly over compacts, and the respective pointwise differences take value in B. If we may assume $v_t \in \mathcal{U}(\mathbf{1} + B)$, write $(\varphi, \mathbb{u}) \simeq_B (\psi, \mathbb{v})$.

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Definition

A cocycle representation $(\theta, \mathbb{x}) : (A, \alpha) \to (\mathcal{M}(B), \beta)$ is called absorbing, if for every cocycle representation $(\varphi, \mathbb{u}) : (A, \alpha) \to (\mathcal{M}(B), \beta)$, we have $(\theta, \mathbb{x}) \oplus (\varphi, \mathbb{u}) \sim_B (\theta, \mathbb{x})$.

Our goal is to generalize the following fundamental theorem from C^*-algebras to C^*-dynamics.

Theorem (Lin, Dadarlat–Eilers)

Let $\varphi, \psi : A \to \mathcal{M}(B)$ be a Cuntz pair of representations, and let $\theta : A \to \mathcal{M}(B)$ be an absorbing representation. Then $[\varphi, \psi] = 0$ in $KK(A, B)$ if and only if $\varphi \oplus \theta \simeq_B \psi \oplus \theta$.

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Theorem (Gabe–S, generalizing Thomsen)

For any actions $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ on separable C^*-algebras, there is an absorbing cocycle representation $(\theta, \mathbb{x}) : (A, \alpha) \to (\mathcal{M}(B), \beta)$.

(The same is true w.r.t. "unitally/nuclearly absorbing" etc.)

Theorem (Gabe–S)

Suppose that

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are three cocycle representations such that the first two form a (α, β)*-Cuntz pair, and* $(\theta, \mathbb{x})$ *is absorbing. Then* $[(\varphi, \mathbb{u}), (\psi, \mathbb{v})] = 0$ *in* $KK^G(\alpha, \beta)$ *if and only if* $(\varphi \oplus \theta, \mathbb{u} \oplus \mathbb{x}) \simeq_B (\psi \oplus \theta, \mathbb{v} \oplus \mathbb{x})$*.*

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Definition

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$$\max_{a \in \mathcal{F}} \|s^*\psi(a)s - \sum_{j=1}^{n} c_j^* \varphi(a) c_j\| \leq \varepsilon$$

and

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This concept simultaneously generalizes two well-studied phenomena. If $G = \{1\}$, then this recovers "weak domination" of ψ by φ as u.c.p. maps. If G is non-trivial but $A = \mathbb{C}$, $B = \mathcal{K}$, $\beta = \mathrm{id}$, $\varphi = \psi = \bullet \cdot \mathbf{1}$, then this recovers weak containment of unitary representations $G \to \mathcal{U}(\ell^2)$.

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Suppose (B, β) is conjugate to $(B \otimes \mathcal{K}, \beta \otimes \mathrm{id}_{\mathcal{K}})$. Choose isometries $t_n \in \mathcal{M}(B)^\beta$ with $\mathbf{1} = \sum_{n \in \mathbb{N}} t_n t_n^*$ in the strict topology. For a sequence of cocycle representations $(\varphi_n, \mathbb{u}^{(n)}) : (A, \alpha) \to (\mathcal{M}(B), \beta)$, we define its direct sum

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If $(\varphi_n, \mathbb{u}^{(n)}) = (\varphi, \mathbb{u})$ is constant, we define the infinite repeat $(\varphi^\infty, \mathbb{u}^\infty)$ accordingly. Up to equivalence via a unitary in $\mathcal{M}(B)^\beta$, none of this depends on the choice of $\{t_n\}$.

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Lemma (Gabe–S; generalizing Voiculescu, Kasparov)

Let $(\varphi,\mathbb{u}),(\psi,\mathbb{v}):(A,\alpha)\to(\mathcal{M}(B),\beta)$ be two cocycle representations. Then $(\psi,\mathbb{v})\precsim(\varphi,\mathbb{u})$ if and only if $(\varphi^\infty,\mathbb{u}^\infty)\sim_B(\varphi^\infty\oplus\psi^\infty,\mathbb{u}^\infty\oplus\mathbb{v}^\infty)$.

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(The proof largely follows the old proofs, but involves lots of additional keeping track of the cocycles in the key steps.)

The existence of (unitally/nuclearly) absorbing cocycle representations is an easy corollary of the following much more general fact. We still assume that A and B are separable and (B, β) is conjugate to $(B \otimes \mathcal{K}, \beta \otimes \mathrm{id}_{\mathcal{K}})$.

Theorem (Gabe–S)

Let $\mathfrak{C}$ be a family of cocycle representations $(A, \alpha) \to (\mathcal{M}(B), \beta)$ that is closed under unitary equivalence via $\mathcal{U}(\mathcal{M}(B)^{\beta})$, and is closed under countable direct sums. Then there exists $(\theta, \mathbb{x}) \in \mathfrak{C}$ such that $(\theta, \mathbb{x}) \oplus (\varphi, \mathbb{u}) \sim_B (\theta, \mathbb{x})$ for all $(\varphi, \mathbb{u}) \in \mathfrak{C}$.

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Proof: The strict topology on $\mathcal{M}(B)$ is metrizable and separable. We equip the set of all cocycle representations $(\varphi, \mathbb{u}) : (A, \alpha) \to (\mathcal{M}(B), \beta)$ with the point-strict topology in the first variable, and the uniform strict topology over compact sets $K \subseteq G$ in the second variable. Since A is separable and G is 2^{nd}-countable, we obtain a separable Polish space.

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Like in the work of Dadarlat–Eilers, one builds on this fact and uses various reduction tricks to show that an equivariant Cuntz pair

$$(\varphi, \mathbb{u}), (\psi, \mathbb{v}) : (A, \alpha) \to (\mathcal{M}(B), \beta)$$

is KK^G-trivial precisely when, after adding an absorbing cocycle representation, they become *operator homotopic* in an appropriate sense.

To end up with the desired statement in the stable uniqueness theorem, one uses all of these facts to carefully set up the application of the following theorem.

Theorem (Gabe–S; generalizing Pedersen, Dadarlat–Eilers)

Let $\delta : G \curvearrowright D$ be an action on a separable unital C^*-algebra. Suppose that $(\varphi, \mathbb{u}) : (D, \delta) \to (D, \delta)$ is a cocycle conjugacy which is ***uniformly norm-homotopic*** *to the identity in the cocycle automorphism group of (D, δ).*

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Similarly to the case $G = \{1\}$, the proof of this fact boils down to the case where $(\varphi, \mathbb{u})$ has uniform norm-distance less than $\frac{1}{2}$ from the identity. Using a measurable cocycle vanishing argument, one first shows that $(\varphi, \mathbb{u})$ is genuinely inner on the double dual (D^{**}, δ) in a specific way.[Thanks, Taka!] Then the rest follows from a Hahn–Banach convexity argument.

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Conjecture (S)

Let Γ be a countable discrete amenable group. Let $\beta : \Gamma \curvearrowright B$ be an outer action on a stable Kirchberg algebra. Let $\alpha : \Gamma \curvearrowright A$ be an action on a separable exact C^*-algebra. Then the canonical map

$$\{\text{coc-hom's } (\varphi, \mathbb{u}) : (A, \alpha) \hookrightarrow (B, \beta)\} / \simeq_B \quad \longrightarrow \quad KK^\Gamma(\alpha, \beta)$$

is a bijection. (Actually this needs to be stated a bit more carefully, but let's not worry about it...)

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Corollary (assuming the above conjecture holds)

Let $\alpha : \Gamma \curvearrowright A$ and $\beta : \Gamma \curvearrowright B$ be outer actions on Kirchberg algebras.

1. *Suppose A and B are stable. Then any invertible element in $KK^{\Gamma}(\alpha, \beta)$ lifts to a cocycle conjugacy.*
2. *Suppose A and B are unital. Then any invertible element in $\kappa \in KK^{\Gamma}(\alpha, \beta)$ with $\kappa([\mathbf{1}_A]_0) = [\mathbf{1}_B]_0 \in K_0(B)$ lifts to a cocycle conjugacy.*

Thank you for your attention!