The dynamical Kirchberg–Phillips theorem

UCLA FAseminar

Gábor Szabó (joint with James Gabe)

May 2022

For this talk, we fix a 2nd-countable, locally compact group G.

Objects of interest: Group actions $\alpha : G \curvearrowright A$ on C^*-algebras.

Goal: Classification up to cocycle conjugacy.

In this vein the special case $G = \{1\}$ should correspond to ordinary C^*-algebra classification, that is, the Elliott program.

In the big picture, this is inspired by the classification of actions of discrete amenable groups on injective factors (Connes, Jones, Ocneanu, ...), which came during/after their classification due to Connes–Haagerup.

Given the history of the Elliott program, the most natural interesting test case for dynamical classification are actions on Kirchberg algebras.† Let us briefly review the classification of these C^*-algebras and then elaborate.

Theorem (Kirchberg–Phillips)

Let A and B be two stable Kirchberg algebras. Then any invertible $x \in KK(A,B)$ lifts to an isomorphism $A \xrightarrow{\cong} B$.

(A few more comments on KK-theory later.)

The actual theorem states that $*$-homomorphisms $A \to B$ are classified by KK-theory up to asymptotic unitary equivalence. The above isomorphism theorem follows as a corollary with the Elliott intertwining machinery.

If both A and B satisfy the UCT, then the existence of an invertible $x \in KK(A,B)$ is equivalent to $K_*(A) \cong K_*(B)$. But this does **not** mean that K-theory classifies $*$-homomorphisms $A \to B$. Hence there is no way around KK-theory to obtain K-theoretic classification.

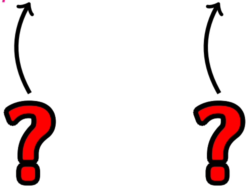
† "Kirchberg algebra" entails: separable simple nuclear and purely infinite

There are various nice results for classifying group actions on Kirchberg algebras, notably by Nakamura, Izumi, Izumi–Matui and others.

These works all have in common that the methodology hinges on the Evans–Kishimoto intertwining technique, which has limitations when the acting group becomes complicated.

Central idea behind new methods: Get rid of said limitations by focusing the classification on the arrows in a suitable category.

$\rightsquigarrow$ What can invariants tell us about *uniqueness* and *existence*?



Definition (S)

Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be two actions on C^*-algebras.

- A cocycle representation is a pair $(\varphi, \mathbb{u}) : (A, \alpha) \to (\mathcal{M}(B), \beta)$, where $\varphi : A \to \mathcal{M}(B)$ is a $*$-homomorphism, $\mathbb{u} : G \to \mathcal{U}(\mathcal{M}(B))$ is a β-cocycle[†], and we have $\mathrm{Ad}(\mathbb{u}_g) \circ \beta_g \circ \varphi = \varphi \circ \alpha_g$ for all $g \in G$.
- If $\varphi(A) \subseteq B$, then $(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta)$ is a cocycle morphism.
- If also $\mathbb{u}(G) \subseteq \mathcal{U}(\mathbf{1} + B)$, then $(\varphi, \mathbb{u})$ is a proper cocycle morphism.
- If φ is an isomorphism, then $(\varphi, \mathbb{u})$ is a (proper) cocycle conjugacy.

Fact

The composition formula $(\psi, \mathbb{v}) \circ (\varphi, \mathbb{u}) := (\psi \circ \varphi, \psi(\mathbb{u})\mathbb{v})$ *turns the (proper) cocycle morphisms into arrows in a category.*

[†] This means $\mathbb{u}$ is strictly continuous and $\mathbb{u}_{gh} = \mathbb{u}_g \beta_g(\mathbb{u}_h)$ for all $g, h \in G$.

Example

For a given action $\beta : G \curvearrowright B$ and a unitary $v \in \mathcal{U}(\mathbf{1}+B)$, the pair

$$\mathrm{Ad}(v) := (\mathrm{Ad}(v), v\beta_\bullet(v)^*)$$

is an inner cocycle morphism on (B, β). Given a cocycle representation $(\varphi, \mathbb{u}) : (A, \alpha) \to (\mathcal{M}(B), \beta)$, the composition is given as

$$\mathrm{Ad}(v) \circ (\varphi, \mathbb{u}) = \big(\mathrm{Ad}(v) \circ \varphi, v\mathbb{u}_\bullet\beta_\bullet(v)^*\big).$$

This gives rise to the notion of unitary equivalence as well as approximate/asymptotic unitary equivalence.

Notation

Write $(\varphi, \mathbb{u}) \ \cong_{\mathrm{u}} (\psi, \mathbb{v})$, if there exists a norm-continuous path $v : [0,\infty) \to \mathcal{U}(\mathbf{1}+B)$ with $v_0 = \mathbf{1}$, $\psi = \lim_{t\to\infty} \mathrm{Ad}(v_t) \circ \varphi$, and $\mathbb{v} = \lim_{t\to\infty} v_t\mathbb{u}_\bullet\beta_\bullet(v_t)^*$ uniformly over compact sets of G.

For the results of today, the following serves as the relevant Elliott intertwining machinery.

Theorem (S)

Let $\alpha : G \curvearrowright A$ *and* $\beta : G \curvearrowright B$ *be actions on separable* C^**-algebras. Suppose that*

$$(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta) \quad \textit{and} \quad (\psi, \mathbb{v}) : (B, \beta) \to (A, \alpha)$$

are two proper cocycle morphisms.

If $(\psi, \mathbb{v}) \circ (\varphi, \mathbb{u}) \approxeq_{\mathrm{u}} \mathrm{id}_A$ *and* $(\varphi, \mathbb{u}) \circ (\psi, \mathbb{v}) \approxeq_{\mathrm{u}} \mathrm{id}_B$*, then* $(\varphi, \mathbb{u})$ *and* $(\psi, \mathbb{v})$ *are* $\approxeq_{\mathrm{u}}$*-equivalent to mutually inverse proper cocycle conjugacies.*

$\rightsquigarrow$ This motivates the study of uniqueness and existence theorems in the given categorical framework.

$\rightsquigarrow$ What promising choices are there among invariants?

The answer for this talk is **equivariant Kasparov theory**. Introducing it properly is outside the scope of this talk, so here comes just a brief preview:

To every pair of actions $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ on separable C^*-algebras, we can assign an abelian group $KK^G(\alpha, \beta)$. This assignment is contravariant in the first variable and covariant in the second. For $G = \{1\}$ one has $KK(\mathbb{C}, A) \cong K_0(A)$ and $KK(\mathcal{C}_0(\mathbb{R}), A) \cong K_1(A)$.

The so-called Kasparov product is a certain pairing map

$$KK^G(\alpha, \beta) \times KK^G(\beta, \gamma) \to KK^G(\alpha, \gamma), \quad (x, y) \mapsto x \otimes y.$$

This allows one to view the G-C^*-algebras as an additive category, with $KK^G(\alpha, \beta)$ being the Hom-set of arrows $\alpha \to \beta$. A KK^G-equivalence is nothing but an isomorphism in this category. Every cocycle morphism $(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta)$ induces an element $KK^G(\varphi, \mathbb{u}) \in KK^G(\alpha, \beta)$ in such a way that composition becomes the Kasparov product.

There's various approaches to (equivariant) KK-theory, each coming with their own advantages and drawbacks. Our detailed methodology is driven by new insights related to the *Cuntz–Thomsen picture*.

In a nutshell, the actions we classify need to be on the one hand *amenable* and on the other hand *sufficiently outer*. Let us now make this precise.

Recent work of Buss–Echterhoff–Willett, Suzuki and Ozawa–Suzuki has provided some long-awaited clarification of what the correct notion of amenability is for actions of locally compact groups on C^*-algebras. There are many equivalent definitions, but the following is useful for us:

Definition (quasicentral approximation property)

Given $\beta : G \curvearrowright B$, we consider $\mathcal{C}_c(G,B)$ as a B-bimodule equipped with the G-action $\bar{\beta}_g(\xi)(h) = \beta_g(\xi(g^{-1}h))$. If μ is the Haar measure, then $\langle f \mid g \rangle = \int_G f(h)^* g(h)\, d\mu(h)$ defines a (compatible) B-valued inner product and induces a norm $\|\cdot\|_2$. (**Note:** $L^2(G,B) := \overline{\mathcal{C}_c(G,B)}^{\|\cdot\|_2}$.)

We say that β is amenable, if there is a net of contractions $\zeta_i \in \mathcal{C}_c(G,B)$ such that

$$\langle \zeta_i \mid \zeta_i \rangle b \to b, \quad \max_{g\in K} \|\zeta_i - \bar{\beta}_g(\zeta_i)\|_2 \to 0, \quad \|b\zeta_i - \zeta_i b\|_2 \to 0$$

for all $b \in B$ and every compact set $K \subseteq G$.

Definition

Recall that the Cuntz algebra $\mathcal{O}_\infty$ can be defined as follows. For a separable ∞-dimensional Hilbert space $\mathcal{H}$, $\mathcal{O}_\infty$ is the universal unital C^*-algebra generated by the range of a linear map $\mathfrak{s}:\mathcal{H}\to\mathcal{O}_\infty$ satisfying the relations $\mathfrak{s}(\eta)^*\mathfrak{s}(\xi)=\langle\xi\mid\eta\rangle\cdot\mathbf{1}$ for all $\xi,\eta\in\mathcal{H}$.

Definition (the model action)

Given a second-countable locally compact group G, we consider the canonical quasi-free action $\gamma: G\curvearrowright\mathcal{O}_\infty$ coming from the choice $\mathcal{H}=\ell^2(\mathbb{N})\hat{\otimes}L^2(G)$ and the formula $\gamma_g\circ\mathfrak{s}=\mathfrak{s}\circ\lambda^\infty_g$, where λ^∞ is the infinite repeat of the left-regular representation.

Definition (Gabe–S)

For an action $\beta: G\curvearrowright B$ on a separable C^*-algebra, we consider $B_{\infty,\beta}$ the β-continuous sequence algebra, and $F_{\infty,\beta}(B)=(B_{\infty,\beta}\cap B')/(B_{\infty,\beta}\cap B^\perp)$ the continuous central sequence algebra. Let us say β is isometrically shift-absorbing, if $F_{\infty,\beta}(B)$ has $(\mathcal{O}_\infty,\gamma)$ as a unital subsystem.

How restrictive are these conditions?

Theorem (Meyer, Ozawa–Suzuki)

Suppose G has the Haagerup property. For any action $\alpha : G \curvearrowright A$ on a separable nuclear C^*-algebra, there is an amenable action $\beta : G \curvearrowright B$ on a stable Kirchberg algebra with $(A, \alpha) \sim_{KK^G} (B, \beta)$.

Note: It is not too difficult to argue that groups fitting into the above theorem have the property that the quotient map $\mathrm{C}^*(G) \to \mathrm{C}^*_r(G)$ is a KK-equivalence. There are hence many counterexamples given by groups with property (T).

Theorem (Ozawa–Suzuki)

For any G, there is an amenable G-action on $\mathcal{O}_2 \otimes \mathcal{K}$.

The following is an application of Kishimoto's work.

Proposition (Izumi–Matui)

If G is discrete and B is a Kirchberg algebra, then $\beta : G \curvearrowright B$ is isometrically shift-absorbing if and only if β is (pointwise) outer.

Warning: This is not true outside the discrete case.

Lemma (Gabe–S)

A flow $\mathbb{R} \curvearrowright B$ on a Kirchberg algebra is isometrically shift-absorbing if and only if it has the Rokhlin property.

Theorem (Pimsner)

The canonical inclusion $(\mathbb{C}, \mathrm{id}) \subseteq (\mathcal{O}_\infty, \gamma)$ is a KK^G-equivalence.

The same follows for $\gamma^{\otimes\infty}$, which is certainly isometrically shift-absorbing. So such actions exhaust all KK^G-classes on Kirchberg algebras.

Side question: Is $\gamma \simeq_{\mathrm{cc}} \gamma^{\otimes\infty}$? (We only know this when G is discrete and amenable.)

The following is what makes these dynamical criteria work well together.

Lemma (Gabe–S)

Let $\beta : G \curvearrowright B$ be an action on a separable C^*-algebra. Then β is isometrically shift-absorbing if and only if there exists a sequence of linear maps $\theta_n : L^2(G,B) \to B$ that is

- *approximately equivariant:* $\theta_n(\bar{\beta}_g(\xi)) \approx \beta_g(\theta_n(\xi))$.
- *approximately B-bilinear:* $\theta_n(b_1\xi b_2) \approx b_1\theta_n(\xi)b_2$.
- *approximately preserving inner products:* $\theta_n(\eta)^*\theta_n(\xi) \approx \langle \xi \mid \eta \rangle_B$.

In a nutshell, being isometrically shift-absorbing means that as a G-equivariant Hilbert B-bimodule, $(B,\beta) \approx (L^2(G,B),\bar{\beta})$.

Amenability of β allows one to perform certain averaging arguments in $L^2(G,B)$. Hence the conjunction of amenability and isometric shift-absorption means that one can "average" inside (B,β).

Before we can state the subsequent results, we need a bit more notation.

Notation

An action $\beta : G \curvearrowright B$ is called strongly stable, if it is conjugate to the action $\beta \otimes \mathrm{id}_{\mathcal{K}} : G \curvearrowright B \otimes \mathcal{K}$.

Note: Strong stability is needed for technical reasons, but is not essential in the final results.

Proposition (Gabe–S)

If B is stable, then every action $\beta : G \curvearrowright B$ is cocycle conjugate to $\beta \otimes \mathrm{id}_{\mathcal{K}} : G \curvearrowright B \otimes \mathcal{K}$. (in general **not** properly!)

As an intermediate result towards classification, we obtain the following dynamical $\mathcal{O}_2$-embedding theorem.

Theorem (Gabe–S)

Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be two actions. Assume that A is separable exact and B is a Kirchberg algebra. Suppose that both α and β are amenable, and that β is strongly stable, isometrically shift-absorbing, and $\beta \simeq_{\mathrm{cc}} \beta \otimes \mathrm{id}_{\mathcal{O}_2}$. There there exists a proper cocycle embedding $(A, \alpha) \to (B, \beta)$, which is unique up to $\approxeq_{\mathrm{u}}$.

Corollary

An action β like above is unique up to proper cocycle conjugacy.

The following is the main technical result of our work:

Theorem (Gabe–S)

Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be two actions. Assume that A is separable exact and B is a Kirchberg algebra. Suppose that both α and β are amenable, and that β is strongly stable and isometrically shift-absorbing. Then:

- *For all $x \in KK^G(\alpha, \beta)$, there is a proper cocycle embedding $(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta)$ with $KK^G(\varphi, \mathbb{u}) = x$ and $\mathbb{u}$ is homotopic to the trivial cocycle.*
- *Let $(\varphi, \mathbb{u}), (\psi, \mathbb{v}) : (A, \alpha) \to (B, \beta)$ be two proper cocycle embeddings with $\mathbb{u}, \mathbb{v}$ homotopic to the trivial cocycle. They are $\approxeq_{\mathrm{u}}$-equivalent if and only if $KK^G(\varphi, \mathbb{u}) = KK^G(\psi, \mathbb{v})$.*

There are similar results for actions on unital C^*-algebras under appropriate assumptions on $x \in KK^G(\alpha, \beta)$.

Recall: For G discrete, "isometrically shift-absorbing"="outer".

Theorem (Gabe–S)

Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be two actions on Kirchberg algebras that are amenable and isometrically shift-absorbing.

- *Suppose α and β are strongly stable. Then every invertible $x \in KK^G(\alpha, \beta)$ lifts to a proper cocycle conjugacy.*
- *Suppose both A and B are stable. Then every invertible $x \in KK^G(\alpha, \beta)$ lifts to a cocycle conjugacy.*
- *Suppose both A and B are unital. Then every invertible $x \in KK^G(\alpha, \beta)$ with $[\mathbf{1}_A]_0 \otimes x = [\mathbf{1}_B]_0$ lifts to a cocycle conjugacy.*

Let us mention some interesting applications:

Corollary

For a Kirchberg algebra A, we get an alternative proof for the uniqueness of Rokhlin flows $\mathbb{R} \curvearrowright A$. In fact, it generalizes to Rokhlin actions $\mathbb{R}^k \curvearrowright A$.

Corollary (generalizing Goldstein–Izumi)

Assume G is discrete amenable. A conjecture of Izumi, asserting that all faithful quasi-free actions $G \curvearrowright \mathcal{O}_\infty$ are cocycle conjugate, is true.

Corollary (Meyer)

A conjecture of Izumi from his 2010 ICM proceedings article is true. (This is a version of the classification theorem for G discrete amenable torsion-free that uses certain bundles in place of KK^G.)

In particular, we recover the main results of recent work of Izumi–Matui for outer actions of poly-$\mathbb{Z}$ groups on Kirchberg algebras. (Inventiones 2021)

Corollary (proved independently by Suzuki)

If G is exact and has the Haagerup property, then there is an amenable action $G \curvearrowright \mathcal{O}_\infty$ that is KK^G-equivalent to $\mathbb{C}$.

Corollary

Let G be discrete exact torsion-free group with the Haagerup property. Let $\mathcal{D}$ be a strongly self-absorbing Kirchberg algebra. Then there is a unique amenable outer G-action on $\mathcal{D}$.

(E.g. $\mathcal{D} = \mathcal{O}_\infty$ or $\mathcal{D} = \mathcal{O}_2$)

Thank you for your attention!